# EE 320 Introductory Mathematical Economics

## Semester 1/2015

## Practice Problem 9

## (Constrained Optimization Problems)

**Question 1:**

The profit obtained by a firm from producing and selling *x* and *y* units of two brands of a commodity is given by

$$P(x,y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$$

a. Find the production levels that maximize profits.
b. A key raw material is rationed so that total production must be restricted to 200 units. Find the production levels that now maximize profits.

**Question 2**

Suppose that Mr. Bean's utility depends on two commodities: $x_1$ and $x_2$. His utility function is given by:

$$U = x_1x_2 + 2x_1 + x_2 \ .$$

Suppose that Mr. Bean's income is $76, and the per-unit prices for the commodities $x_1$ and $x_2$ are $2 and $3, respectively.

a. Determine the values $x_1^*$ and $x_2^*$ that maximize Mr. Bean's utility, given that Mr.Bean spends all of his income on these two commodities.

b. Show the second-order sufficient condition for the constrained utility maximization.

**Question 3:**

Suppose that a monopolistic firm sells a single product in three separate markets (say, three different countries), and therefore faces three different demand functions:

$$P_1 = 32 - 1.5Q_1$$
$$P_2 = 68 - 3Q_2$$
$$P_3 = 40 - 2Q_3$$

The total cost function is given by:

$TC = 60 + 20Q$ where $Q = Q_1 + Q_2 + Q_3$

a. Suppose that there is a shortage of a key raw material so that the total production must be restricted to $\overline{Q} = 12$ units. Find the profit-maximizing output levels of $Q_1$, $Q_2$, and $Q_3$, and determine the maximum profit.
b. Use the bordered Hessian matrix to verify that the second-order sufficient condition is met.

**Question 4**

A firm produces and sells two commodities. By selling *x* tons of the first commodity the firm gets a price per ton given by $p = 96 - 4x$. By selling *y* tons of the other commodity the price per ton is given by $q = 84 - 2y$.

The cost of producing and selling *x* tons of the first commodity and *y* tons of the second is given by:

$$C(x, y) = 2x^2 + 2xy + y^2 \ .$$

a. Show that the firm's profit function is:
   $P(x,y) = -6x^2 - 3y^2 - 2xy + 96x + 84y$
b. Compute the first-order partial derivatives of profit (*P*) with respect to *x* and *y* , and find its only stationary point.
c. Suppose the production causes pollution, and that the authorities for this reason require the firm the produce only 11 tons in total. Solve the firm's maximization problem in this case. Verify that the production restrictions do reduce the maximum possible value of $P(x,y)$ .

**Question 5:**

Given the utility maximization problem

$$Max_{x,y} U(x,y) = \sqrt{x} + y$$

subject to $x + 4y = 100$

a. Find the quantities demanded of the two goods using the Lagrange method. Determine the maximum utility level and the value of the Lagrange multiplier.
b. Suppose income increases from 100 to 101. What is the exact increase in the optimal value of U(x, y)? Compare with the value found in (a) for the Lagrange multiplier.
c. Suppose we change the budget constraint to $px + qy = m$ , but keep the same utility function. Derive the quantities demanded of the two goods if $m > \frac{q^2}{4p}$ .

**Question 6**:

A firm has an order of 10,000 units of its product and has two plants at which to manufacture these units. Let $q_1$ be the number of units to be produced at the first plant and $q_2$ denote the number to be manufactured at the second plant. Suppose that the cost function is given by
$C = 48q_1^3 + 3\,q_2^3 + 25,000$.

a. Use the method of Lagrange multipliers to determine how many units should be produced at each plant to minimize this cost function.
b. Confirm your result by checking the second-order condition.

**Question 7:** *Consumption-Leisure choice*
An individual has a Cobb–Douglas utility function $U(M, L) = AM^aL^b$, where m is income and l is leisure. Here A, a, and b are positive constants, with a + b ≤ 1. A total of $T_0$ hours are allocated between work ($W$) and leisure ($L$), so that $W + L = T_0$. If the hourly wage is w, then $M = \overline{w}W$ , and the individual's problem is

$$max_{M,L}\ \ AM^aL^b\ st.\ \ M/\overline{w} + l = T_0$$

a. Solve for optimal leisure that this individual will choose.
b. Confirm your result by checking the second-order condition.
c. How does the change in hourly wage affect the decision to work?

**Question 8**: Intertemporal consumption problem

Consider a consumer who lives in two periods of time. In the first period, the consumer earns income $Y_0$. This income can be allocated for consumption in the current period ($C_0$) or saving ($S$) for the future use. The amount of saving that the consumer chooses can be used in the subsequent period for consumption. Every unit of saving would yield this consumer $(1+r)$ of proceeds that can be used for the consumption in the future ($C_1$) .

Let $C_0$ be the amount of today's consumption, $C_1$ be the amount of future's consumption, $S$ be the amount of savings, and $1+r$ be gross yield from saving. The problem that this consumer faces is to choose for different level of consumption between the two periods, maximizing the following utility-based criteria function given by,

$U = ln(C_0) + \beta ln(C_1)$ subject to $C_1 = (1+r)(Y_0 - C_0)$
where $0 < \beta < 1$.

Consider the problem.

a. Solve for the optimal consumption in the two periods using the LaGrange multiplier method.
b. Confirm your result in a by checking the second-order condition.
c. What happen to $C_0$ if $\beta$ increases. Explain the intuition of your result.
d. How does the interest rate affect the amount of saving? Explain the intuition of your result.

**Question 9**: Cost minimization problem.

Consider a representative firm with the production function taking the form: $Q = K^a + L^a$, where K is the capital, L is the labor input, Q is the amount of output, and $a \in (0, 1)$. This firm purchases capital and labor for the market at the price per unit of $r$ and $w$, respectively. Consider the following problem

a. Does the production function exhibit returns to scale technology? If yes, state the type of returns to scale.
b. Use the second-order derivative matrix and show that the production function is concave.
c. Derive the cost-minimizing bundle of factor inputs.
d. Confirm your result in "c" that your proposed solution is the least cost combination of factor inputs.
e. Show the expression for LaGrange multiplier.
f. Derive the minimized cost function.
g. Show that marginal cost is the LaGrange multiplier.
h. Show that the minimized cost function is concave in the prices of factor inputs. That is, show that the Hessian of minimized cost function is negative definite.

**Question 10**:

Solve for the following constrained optimization problem.

a. $min\ \ x^2 + y^2 + z^2\ \ st.\ \ x + y + z = 1$
b. $max\ U(x, y, L) = 2xy(24 - L)\ \ st.\ p_x x + p_y y\ = wL$