FN211: Lecture Note 6

Basic Matrices

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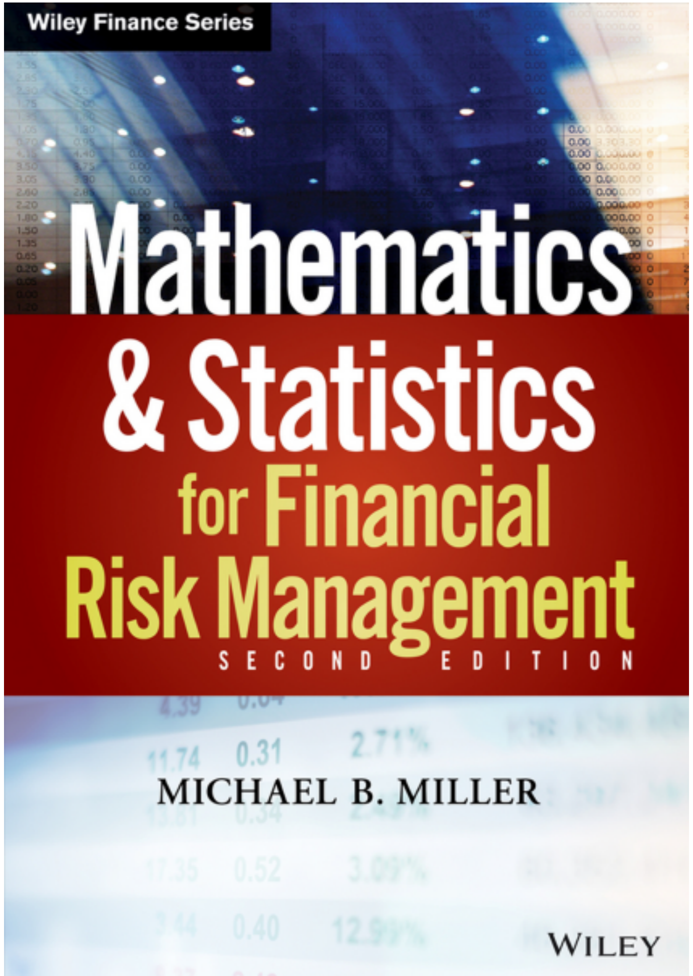

Outline

- Introduction and Types of Matrices
- Matrix Operations
- Inverse of a Matrix
- Application

Reading: Chapter 8

AND: bit.ly/211mattext

Introduction and Types of Matrices

(1) Column matrix or vector

(2) Row matrix or vector

(3) Rectangular matrix

(4) Square matrix

(5) Diagonal matrix

(6) Unit or Identity matrix - I

(7) Null (zero) matrix – 0

(8) Triangular matrix

(9) Scalar matrix

Matrices - Introduction

- Matrix algebra has at least two advantages:
- Reduces complicated systems of equations to simple expressions
- Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices - Introduction

- **Properties:**
  - A specified number of rows and a specified number of columns
  - Two numbers (rows x columns) describe the dimensions or size of the matrix.

Examples:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Matrices - Introduction

- A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

- e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = \quad {}_{m}\mathbf{A}^{n} \quad \begin{bmatrix} a_{11} & a_{12}\cdots & a_{ij} & a_{in} \\ a_{21} & a_{22}\cdots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

Types of Matrices

1.Column matrix or vector

- The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

2. Row matrix or vector

- Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

Types of Matrices

3. Rectangular matrix

- Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

4. Square matrix

- The number of rows is equal to the number of columns (a square matrix A m x m has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Types of Matrices

5. Diagonal matrix

- A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. a_{ij} =0 for all i not equal j

a_{ij} = 0 for some or all $i = j$

6. Unit or Identity matrix - I

- A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. a_{ij} =0 for all i not equal j

a_{ij} = 1 for some or all $i = j$

Types of Matrices

7. Null (zero) matrix - 0

- All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0$$ For all i,j

8. Triangular matrix

- A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Types of Matrices

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

Types of Matrices

9. Scalar matrix

- A diagonal matrix whose main diagonal elements are equal to the same scalar
- A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all i not equal j

$a_{ij} = a$ for all $i = j$

Matrix Operations

(1) Equality of matrices

(2) Addition and subtraction of matrices

(3) Scalar multiplication of matrices

(4) Multiplication of matrices

(5) Transpose of a Matrix

(1) EQUALITY OF MATRICES

- Two matrices are said to be equal only when all corresponding elements are equal
- Therefore their size or dimensions are equal as well

A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ **B** = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ **A = B**

(1) EQUALITY OF MATRICES

- Some properties of equality:
- IIf **A** = **B**, then **B** = **A** for all **A** and **B**
- IIf **A** = **B**, and **B** = **C**, then **A** = **C** for all **A**, **B** and **C**

A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ **B** = $\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

If **A** = **B** then $a_{ij} = b_{ij}$

(2) ADDITION AND SUBTRACTION OF MATRICES

- The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

- Matrices of different sizes cannot be added or subtracted
- Commutative Law: **A** + **B** = **B** + **A**
- Associative Law: **A** + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

Example: Sales

The A-Plus auto parts store chain has two outlets, one in Vancouver and one in Quebec. Among other things, it sells wiper blades, windshield cleaning fluid, and floor mats. The monthly sales of these items at the two stores for two months are given in the following tables:

January Sales

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

February Sales

	Vancouver	Quebec
Wiper Blades	23	12
Cleaning Fluid (bottles)	8	12
Floor Mats	4	5

Use matrix arithmetic to calculate the change in sales of each product in each store from January to February.

(2) ADDITION AND SUBTRACTION OF MATRICES

- $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$ (where $-\mathbf{A}$ is the matrix composed of $-a_{ij}$ as elements)

(3) SCALAR MULTIPLICATION OF MATRICES

- Matrices can be multiplied by a scalar (constant or single element)

- Let k be a scalar quantity; then kA = Ak

Ex. If k=4 and $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$, then

Example: Sales (2)

The revenue generated by sales in the Vancouver and Quebec branches of the A-Plus auto parts store (see Example 2) was as follows:

January Sales in Canadian Dollars

	Vancouver	Quebec
Wiper Blades	140.00	105.00
Cleaning Fluid	30.00	36.00
Floor Mats	96.00	48.00

If the Canadian dollar was worth $0.65 U.S. at the time, compute the revenue in U.S. dollars.

(3) SCALAR MULTIPLICATION OF MATRICES

- Properties:

- k (**A** + **B**) = k**A** + k**B**
- (k + g)**A** = k**A** + g**A**
- k(**AB**) = (k**A**)**B** = **A**(k)**B**
- k(g**A**) = (kg)**A**

(4) MULTIPLICATION OF MATRICES

- The product of two matrices is another matrix
- Two matrices **A** and **B** must be **conformable** for multiplication to be possible i.e. the number of columns of **A** must equal the number of rows of **B**

Example:

A x **B** = **C**

(1x3) (3x1) (1x1)

B x **A** =

(2x1) (4x2)

A x **B** =

(6x2) (6x3)

A x **B** =

(2x3) (3x2)

(4) MULTIPLICATION OF MATRICES

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

- Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

(4) MULTIPLICATION OF MATRICES

- Example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} =$

Remember also: IA = A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Example: Revenue

The A-Plus auto parts store mentioned in examples in the previous section had the following sales in its Vancouver store:

	Vancouver
Wiper Blades	20
Cleaning Fluid (bottles)	10
Floor Mats	8

The store sells wiper blades for \$7.00 each, cleaning fluid for \$3.00 per bottle, and floor mats for \$12.00 each. Use matrix multiplication to find the total revenue generated by sales of these items.

Example: Revenue (2)

January sales at the A-Plus auto parts stores in Vancouver and Quebec are given in the following table.

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

The usual selling prices for these items are \$7.00 each for wiper blades, \$3.00 per bottle for cleaning fluid, and \$12.00 each for floor mats. The discount prices for A-Plus Club members are \$6.00 each for wiper blades, \$2.00 per bottle for cleaning fluid, and \$10.00 each for floor mats. Use matrix multiplication to compute the total revenue at each store, assuming first that all items were sold at the usual prices, and then that they were all sold at the discount prices.

(4) MULTIPLICATION OF MATRICES

- Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:
  1. **AI** = **IA** = **A**
  2. **A**(**BC**) = (**AB**)**C** = **ABC** - (associative law)
  3. **A**(**B**+**C**) = **AB** + **AC** - (first distributive law)
  4. **(A+B)C = AC + BC** - (second distributive law)

Caution!

1. **AB** not generally equal to **BA**, **BA** may not be conformable
2. If **AB** = **0**, neither **A** nor **B** necessarily = **0**
3. If **AB** = **AC**, **B** not necessarily = **C**

(4) MULTIPLICATION OF MATRICES

- **AB** not generally equal to **BA, BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS =$$

$$ST =$$

(4) MULTIPLICATION OF MATRICES

- If **AB** = **0**, neither **A** nor **B** necessarily = **0**

Example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} =$$

(5) TRANSPOSE OF A MATRIX

- If : $$\underset{2x3}{A} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

- Then transpose of A, denoted A^T is: $$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a^T_{ji}$$ For all *i* and *j*

(5) TRANSPOSE OF A MATRIX

- To transpose:
- Interchange rows and columns
- The dimensions of A^T are the reverse of the dimensions of A

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

2 x 3

$$A^T = {}_3A^{T^2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

3 x 2

(5) TRANSPOSE OF A MATRIX

- Properties of transposed matrices:

1. $(A+B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(kA)^T = kA^T$
4. $(A^T)^T = A$

(5) TRANSPOSE OF A MATRIX

- Properties of transposed matrices:

1. $(A+B)^T = A^T + B^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} =$$

(5) TRANSPOSE OF A MATRIX

- Properties of transposed matrices:

  2. $(AB)^T = B^T A^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} =$$

Inverse of a Matrix

INVERSE OF A MATRIX

- Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example: k=7 the inverse of k or $k^{-1} = 1/k = 1/7$

- **Division of matrices is not defined** since there may be **AB** = **AC** while **B** is not equal to **C**.
- Instead matrix inversion is used.
- The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

INVERSE OF A MATRIX

- The inverse of a square matrix, A, if it exists, is the unique matrix A^{-1} where:

$$AA^{-1} = A^{-1}A = I$$

Example:

$$A =_2 A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} =$$

INVERSE OF A MATRIX

- Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

Note:

- A square matrix that has an inverse is called a nonsingular matrix
- A matrix that does not have an inverse is called a singular matrix
- Square matrices have inverses except when the determinant is zero
- When the determinant of a matrix is zero, the matrix is singular

INVERSE OF A MATRIX

- The inverse of a square matrix: $A^{-1} = \dfrac{adjA}{|A|}$

  (A) Adjoint of matrix

  (B) Determination of matrix

TO BE CONTINUED ...

(A) ADJOINT MATRIX

- The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

- The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

- A minor of **A**, m_{ij} , is the determinant of the submatrix when deleting row and column (*i* and *j*), respectively.
- Then, m_{ij} is the minor of the element a_{ij} in **A**.

(A) ADJOINT MATRIX

MINORS

- If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.
- The determinant of such a submatrix is called a minor of **A.**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining

2 x 2 submatrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

(A) ADJOINT MATRIX

MINORS

- Therefore the minor of a_{12} is: $$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

- And the minor for a_{13} is: $$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(A) ADJOINT MATRIX

- *Example:* Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, compute its minors.

(A) ADJOINT MATRIX

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as: $$C_{ij} = (-1)^{i+j} m_{ij}$$

- When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i*+*j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i = 1, j = 1) = (-1)^{1+1} m_{11} = +m_{11}$$
$$c_{12}(i = 1, j = 2) = (-1)^{1+2} m_{12} = -m_{12}$$
$$c_{13}(i = 1, j = 3) = (-1)^{1+3} m_{13} = +m_{13}$$

(A) ADJOINT MATRIX

COFACTORS

Example: Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, compute its cofactors.

(A) ADJOINT MATRIX

- The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

Example: Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, compute the adjoint matrix.

(B) DETERMINANT OF A MATRIX

- To compute the inverse of a matrix, the determinant is required
- Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or **|A|**

If $$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

(B) DETERMINANT OF A MATRIX

- If **A** = [**A**] is a single element (1x1), then the determinant is defined as the value of the element

- Then |**A**| =det **A** = a_{11}

- If **A** is (n x n), its determinant may be defined in terms of order (n-1) or less.

(B) DETERMINANT OF A MATRIX

- The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

- The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

- (It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

(B) DETERMINANT OF A MATRIX

Example: For matrix 2x2:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

What if $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$?

Has cofactors : $c_{11} = m_{11} = |a_{22}| = a_{22}$

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of A is: $|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$

(B) DETERMINANT OF A MATRIX

Example: Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, compute its determinant.

Note: ADJOINT MATRIX AND DETERMINANT

- It can be shown that: A(adj A) = (adjA) A = |A| I

Example: $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ *... show the above expression is true.*

INVERSE OF A MATRIX - USING THE ADJOINT MATRIX

- From $AA^{-1} = A^{-1} A = I$,

  and $A(\text{adj } A) = (\text{adj}A) A = |A| I$,

  then, $$A^{-1} = \frac{adjA}{|A|}$$

INVERSE OF A MATRIX - USING THE ADJOINT MATRIX

Example: Find the inverse of the following matrices:

(1) $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

Application

(1) Solving Systems of Equations

(2) Input-Output Models

Solving Systems of Equations

Solve a System of *n* Linear Equations in *n* Unknowns

Having used systems of equations and row reduction to find matrix inverses, we will now use matrix inverses to solve systems of equations. Recall that, at the end of the previous section, we saw that a system of linear equations could be written in the form

$$AX = B$$

where A is the coefficient matrix, X is the column matrix of unknowns, and B is the column matrix of right-hand sides.

Example

The system of linear equations

$$\begin{aligned} 2x \qquad + z &= 9 \\ 2x + y - z &= 6 \\ 3x + y - z &= 9 \end{aligned}$$

Solve a System of *n* Linear Equations in *n* Unknowns

From: $AX = B$

The object is to solve for the matrix *X* of unknowns:

Solve a System of *n* Linear Equations in *n* Unknowns

Example Solve the following three systems of equations.

a.
$$\begin{aligned} 2x \quad\quad\; + z &= 1 \\ 2x + y - z &= 1 \\ 3x + y - z &= 1 \end{aligned}$$

b.
$$\begin{aligned} 2x \quad\quad\; + z &= 0 \\ 2x + y - z &= 1 \\ 3x + y - z &= 2 \end{aligned}$$

c.
$$\begin{aligned} 2x \quad\quad\; + z &= 0 \\ 2x + y - z &= 0 \\ 3x + y - z &= 0 \end{aligned}$$

Solve a System of *n* Linear Equations in *n* Unknowns

Nutrition A four-ounce serving of Campbell's® Pork & Beans contains 5 grams of protein and 21 grams of carbohydrates.[16] A typical slice of "lite" rye bread contains 4 grams of protein and 12 grams of carbohydrates.

I am planning a meal of "beans-on-toast" and I want it to supply 20 grams of protein and 80 grams of carbohydrates. How should I prepare my meal?

Resource Allocation You manage an ice cream factory that makes three flavors: Creamy Vanilla, Continental Mocha, and Succulent Strawberry. Into each batch of Creamy Vanilla go two eggs, one cup of milk, and two cups of cream. Into each batch of Continental Mocha go one egg, one cup of milk, and two cups of cream. Into each batch of Succulent Strawberry go one egg, two cups of milk, and one cup of cream. Your stocks of eggs, milk, and cream vary from day to day. How many batches of each flavor should you make in order to use up all of your ingredients if you have the following amounts in stock?

a. 350 eggs, 350 cups of milk, and 400 cups of cream
b. 400 eggs, 500 cups of milk, and 400 cups of cream
c. A eggs, B cups of milk, and C cups of cream

Input-Output Models

Input-Output Models

Inputs and Pollutants' Output \ Output Industries	Industry 1	Industry 2	Pollution Abatement	Final Demand	Total Output
Industry 1	X_{11}	X_{12}	X_{1pa}	Y_1	X_1
Industry 2	X_{21}	X_{22}	X_{2pa}	Y_2	X_2
Physical Output of Pollutant	X_{1p}	X_{2p}	$-X_{pa}$	Y_p	X_p
Primary Inputs	V_1	V_2	V_{pa}	V_y	V
Total Inputs	X_1	X_2	X_{pa}	Y	X

Input-Output Models

Total supply from Sector 1 = Total demand for Sector 1 products

$$x_1 = \underset{\substack{\uparrow \\ \text{Coal required by Sector 1}}}{0.50x_1} + \underset{\substack{\uparrow \\ \text{Coal required by Sector 2}}}{0.25x_2} + \underset{\substack{\uparrow \\ \text{External demand for coal}}}{7000}$$

Total supply from Sector 2 = Total demand for Sector 2 products

$$x_2 = \underset{\substack{\uparrow \\ \text{Electricity required by Sector 1}}}{0.10x_1} + \underset{\substack{\uparrow \\ \text{Electricity required by Sector 2}}}{0.25x_2} + \underset{\substack{\uparrow \\ \text{External demand for electricity}}}{14{,}000}$$

Input-Output Models

This is a system of two linear equations in two unknowns:

$$x_1 = 0.50x_1 + 0.25x_2 + 7000$$
$$x_2 = 0.10x_1 + 0.25x_2 + 14{,}000$$

We can rewrite this system of equations in matrix form as follows:

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Production}} = \underbrace{\begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Internal demand}} + \underbrace{\begin{bmatrix} 7000 \\ 14{,}000 \end{bmatrix}}_{\text{External demand}}$$

In symbols,

$$X = AX + D$$

How to solve for production vector?

Input-Output Models

Note:

$$A = \begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix}$$

is called the **technology matrix.** The entries of the technology matrix have the following meanings:

$$a_{11} = \text{units of Sector 1 needed to produce one unit of Sector 1}$$
$$a_{12} = \text{units of Sector 1 needed to produce one unit of Sector 2}$$
$$a_{21} = \text{units of Sector 2 needed to produce one unit of Sector 1}$$
$$a_{22} = \text{units of Sector 2 needed to produce one unit of Sector 2}$$

Input-Output Models: Example (1)

Example **Petroleum and Natural Gas**

Consider two sectors of the U.S. economy: crude petroleum and natural gas (*crude*) and petroleum refining and related industries (*refining*). According to government figures,* in 1998 the crude sector used $27,000 million worth of its own products and $750 million worth of the products of the refining sector to produce $87,000 million worth of goods (crude oil and natural gas). The refining sector in the same year used $59,000 million worth of the products of the crude sector and $15,000 million worth of its own products to produce $140,000 million worth of goods (refined oil and the like). What was the technology matrix for these two sectors? What was left over from each of these sectors for use by other parts of the economy or for export?

Input-Output Models: Example (2)

Example **Rising Demand**

Suppose that external demand for refined petroleum rises to \$200,000 million, but the demand for crude remains \$1000 million (as in Example 1). How do the production levels of the two sectors considered in Example 1 have to change?

Input-Output Models: Example (3-4)

- ***Campus Food*** The two campus cafeterias, the Main Dining Room and Bits & Bytes, typically use each other's food in doing business on campus. One weekend, the input-output table was as follows.[32]

To	**Main DR**	**Bits & Bytes**
From **Main DR**	\$10,000	\$20,000
Bits & Bytes	5000	0
Total Output	50,000	40,000

Given that the demand for food on campus last weekend was \$45,000 from the Main Dining Room and \$30,000 from Bits & Bytes, how much did the two cafeterias have to produce to meet the demand last weekend?

- ***Plagiarism*** Two student groups at Enormous State University, the Choral Society and the Football Club, maintain files of term papers that they write and offer to students for research purposes. Some of these papers they use themselves in generating more papers. In order to avoid suspicion of plagiarism by faculty members (who seem to have astute memories), each paper is given to students or used by the clubs only once (no copies are kept). The number of papers that were used in the production of new papers last year is shown in the following input-output table:

To	**Choral Soc.**	**Football Club**
From **Choral Soc.**	20	10
Football Club	10	30
Total Output	100	200

Given that 270 Choral Society papers and 810 Football Club papers will be used by students outside of these two clubs next year, how many new papers do the two clubs need to write?

BACHELOR OF ECONOMICS
THAMMASAT UNIVERSITY

Question?