Exercise 5: Relation & Function

1 Part I

1. Let $A = \{1, 2, 3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A. Define a relation r from $\mathcal{P}(A)$ to $\mathbb{Z}$ as

$$r = \{(x, y) \in \mathcal{P}(A) \times \mathbb{Z} | y = \text{ the number of elements in } x\}.$$

   (a) List all the elements in r.

   (b) Is r a function? If so, find the domain, co-domain, and range of r.

2. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define a function $f : X \to Y$ by $f = \{(1, a), (2, a), (3, c)\}$.

   (a) Find the domain of f, co-domain of f,and range of f.

   (b) What is the inverse image of a?

   (c) What is $f(2)$?

   (d) Draw the arrow diagram of f.

3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (2x + 1, \tfrac{1-y}{2}) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

   (a) Is H one-to-one? Prove or give a counterexample.

   (b) Is H onto? Prove or give a counterexample.

   (c) Is H bijective? If so, find H^{-1}, the inverse function of H.

4. Let $f : (0, \infty) \to \mathbb{R}$ defined by

$$f = \begin{cases} |x|, & x \in (0, 1] \\ x^2 & x \in (1, 3] \\ 6 + |x|, & x \in (3, \infty). \end{cases}$$

   (a) Is f one-to-one? Prove or give a counterexample.

   (b) Is f onto? Prove or give a counterexample.

   (c) Is f bijective? If so, find the inverse function of f.

5. Find the largest sets D and S such that the function $f : D \to S$ defined by $f(x) = \frac{2}{x-1}$ is an onto function.

6. Let $f : \mathbb{R} - \{1\} \to S$ be a *bijective* function defined by $f(x) = \frac{x+1}{x-1}$.

   (a) Determine the set S.

   (b) Determine the inverse function f^{-1}.

   (c) Compute $f \circ f$ and $f \circ f^{-1}$.

7. Graph the function $f:\mathbb{R}\to\mathbb{Z}^+\cup\{0\}$, $f(x)=\lfloor x\rfloor-\lceil x\rceil$ on the closed interval $[-4,4]$.

8. If $f:X\to Y$ and $g:Y\to Z$ are functions and $g\circ f:X\to Z$ is onto, must both f and g be onto? Prove or give a counterexample.

9. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f\circ g$, $g\circ f$, and determine whether or not $f\circ g=g\circ f$ for the given formulas for f and g. Compute $(f\circ g)(2)$ and $(g\circ f)(2)$.

   (a) $f(x)=\frac{x}{\sqrt{x^2+1}}$, $\quad g(x)=x^3+1$.

   (b) $f(x)=x^5$, $\quad g(x)=x^{1/5}$.

10. Let f and g be functions defined by $f(x)=\sqrt{2-x}, x\geq 2$ and

$$g=\begin{cases} x^2-1, & x\in(-\infty,-1] \\ \frac{1}{x}, & x\in(-1,\infty). \end{cases}$$

   Find $g\circ f$ and its domain.

2 Part II

1. Let $X=\{1,2,3\}$ and $Y=\{a,b,c,d\}$. Define relations
   $f:X\to Y$ by $f=\{(1,a),(2,a),(3,c)\}$,
   $g:X\to Y$ by $g=\{(1,a),(3,c)\}$, and
   $h:X\to Y$ by $h=\{(1,a),(2,a),(3,b),(3,c)\}$.

   (a) Draw the arrow diagrams of f, g, and h.

   (b) Show that f is a function, but g and h are not functions.

   (c) Find the domain of f, co-domain of f,and range of f.

   (d) What is the inverse image of a for the function f?

   (e) What is $f(3)$?

2. Let $A=\{-1,1,2,4\}$ and $B=\{1,2\}$ and define relations R and S from A to B as follows: For all $(x,y)\in A\times B$,

$$x\ R\ y \Leftrightarrow |x|=|y|.$$

   and

$$x\ S\ y \Leftrightarrow x-y \text{ is even.}$$

   State explicitly the sets $A\times B$, R, S, $R\cup S$, and $R\cap S$.

3. Let $A=\{1,2,3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A, and

$$X=\{x\in\mathcal{P}(A)|\ x\cap\{1\}\neq\emptyset\}.$$

   Define a relation r from X to $\mathbb{Z}$ as

$$r=\{(x,y)\in X\times\mathbb{Z}|\ y=\text{ the number of elements in } x\}.$$

   (a) List all elements in X.

(b) Draw an arrow diagram of r .

(c) Is r a function? If so, find the domain, co-domain, and range of r.

4. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (x + 1, 2y - 3) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

5. Define functions $f_1 : [0, 2) \to \mathbb{R}$ as

$$f_1(x) = x^2$$

and define $f_2 : [2, \infty) \to \mathbb{R}$ as

$$f_2(x) = 3x - 2.$$

Let $F : [0, \infty) \to [0, \infty)$ be a function defined by using f_1 and f_2:
$F(x) = f_1(x), \quad$ for $\quad x \in [0, 2)$, and $F(x) = f_2(x), \quad$ for $\quad x \in [2, \infty)$. That is,

$$F(x) = \begin{cases} x^2, & x \in [0, 2) \\ 3x - 2, & x \in [2, \infty). \end{cases}$$

(a) Find the domain, co-domain, and range for each of the functions f_1, f_2, and F.

(b) Construct the composite functions $f_1 \circ f_2$, $f_2 \circ f_1$, and $f_1 \circ F$ (if possible). Determine the domains and ranges for these composite functions.

(c) Are f_1 and f_2 injective? Explain.

(d) Is the function F bijective? If so, find the **inverse function** of F.

6. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

(a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$

(b) $f(x) = x^5, \quad g(x) = x^{1/5}.$

7. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.

(a) Compute $f \circ f$ and determine its domain.

(b) Determine whether f is bijective. If so, find the inverse function f^{-1} and $f \circ f^{-1}$.

3 Part III

1. Let $X = \{2, 3, 4\}$ and $Y = \{1, 5\}$. Define relations
   $f : X \to Y$ by $f = \{(x, y) \in X \times Y | x > y\}$,
   $g : X \to Y$ by $g = \{(x, y) \in X \times Y | x = 2y\}$, and
   $h : X \to Y$ by $h = \{(x, y) \in X \times Y | y = x^2\}$.
   (a) List all the elements of the Cartesian product $X \times Y$.
   (b) List all the elements of the relations f, g, and h.
   (c) Determine which of the relations f, g, h is a function from X to Y. Explain your answer.
   (d) What is the inverse image of 1 under f? What is the inverse image of 5 under f?

2. Let $A = \{1, 2, 3\}$ and let $\mathcal{P}(A)$ be the set of all subsets of the set A. Define a relation r and s as
   $$r = \{(x, y) \in A \times \mathcal{P}(A) \mid x = \text{ the number of elements in } y\},$$
   $$s = \{(u, v) \in \mathcal{P}(A) \times A \mid (v, u) \in r\}.$$
   (a) Draw arrow diagrams of r and s.
   (b) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.
   (c) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.

3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:
   $$H(x, y) = (y, x - 2) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$
   (a) Is H one-to-one? Prove or give a counterexample.
   (b) Is H onto? Prove or give a counterexample.
   (c) Is H bijective? If so, find H^{-1}, the inverse function of H.

4. Define functions f and g as follows:
   $f = \{(1, 10), (3, 30), (5, 50)\}$ and
   $g = \{(10, k), (20, \ell), (30, m), (40, n), (50, t)\}$.
   (a) Determine the domain and range for each of functions f and g.
   (b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.
   (c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.

5. Define
   $$g(x) = \frac{1}{\sqrt{x+1}} + 1 \qquad \text{and} \qquad F(x) = \begin{cases} x^2 - 1, & x \in [-3, 1) \\ 2 - x, & x \in [1, 4]. \end{cases}$$
   (a) Find the domain and range for each of the functions g and F.
   (b) Construct the composite functions $F \circ g$, and $g \circ F$ (if possible). Determine the domains for these composite functions.
   (c) Is the function F bijective? If so, find the **inverse function** of F. Justify your answer.