

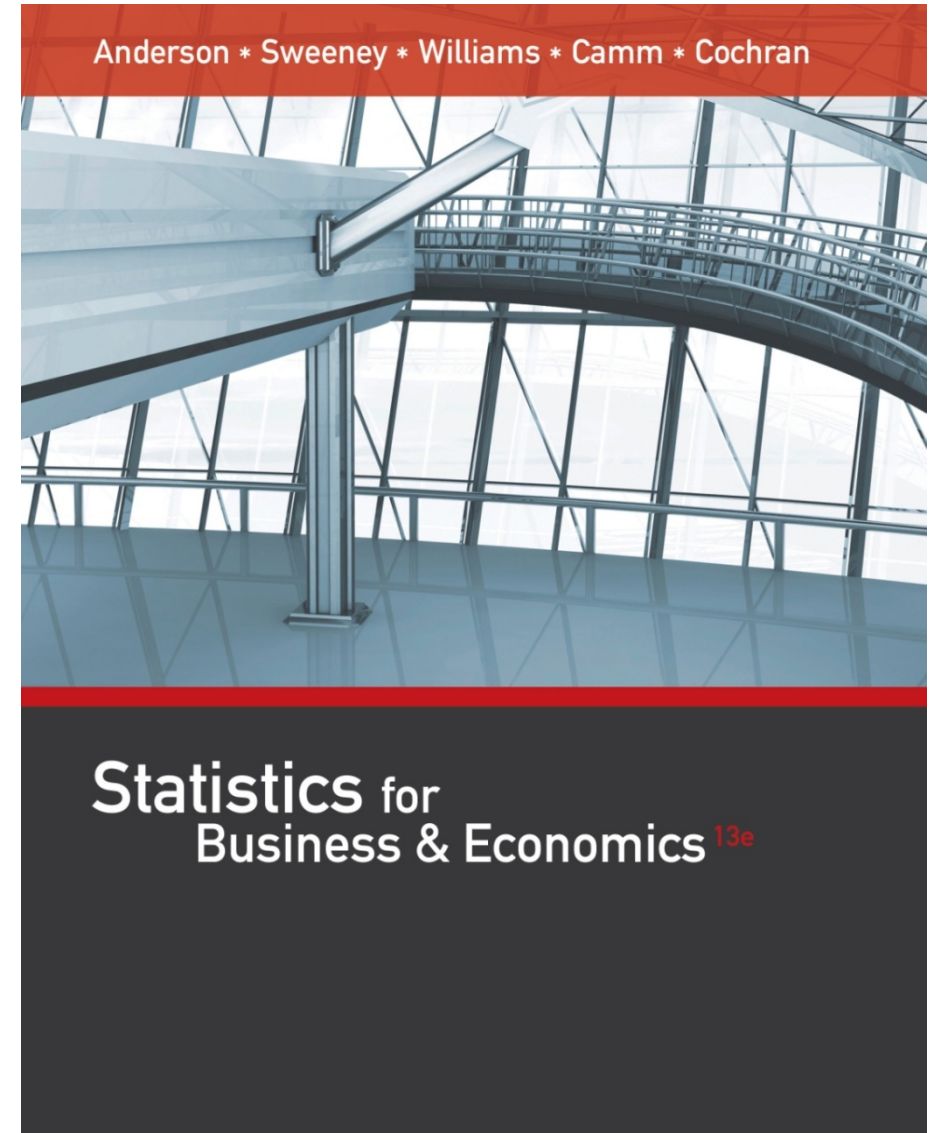
Statistics for Business and Economics (13e)

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Chapter 4

Introduction to Probability

- Random Experiments, Counting Rules, and Assigning Probabilities
- Events and Their Probability
- Some Basic Relationships of Probability
- Conditional Probability
- Bayes' Theorem

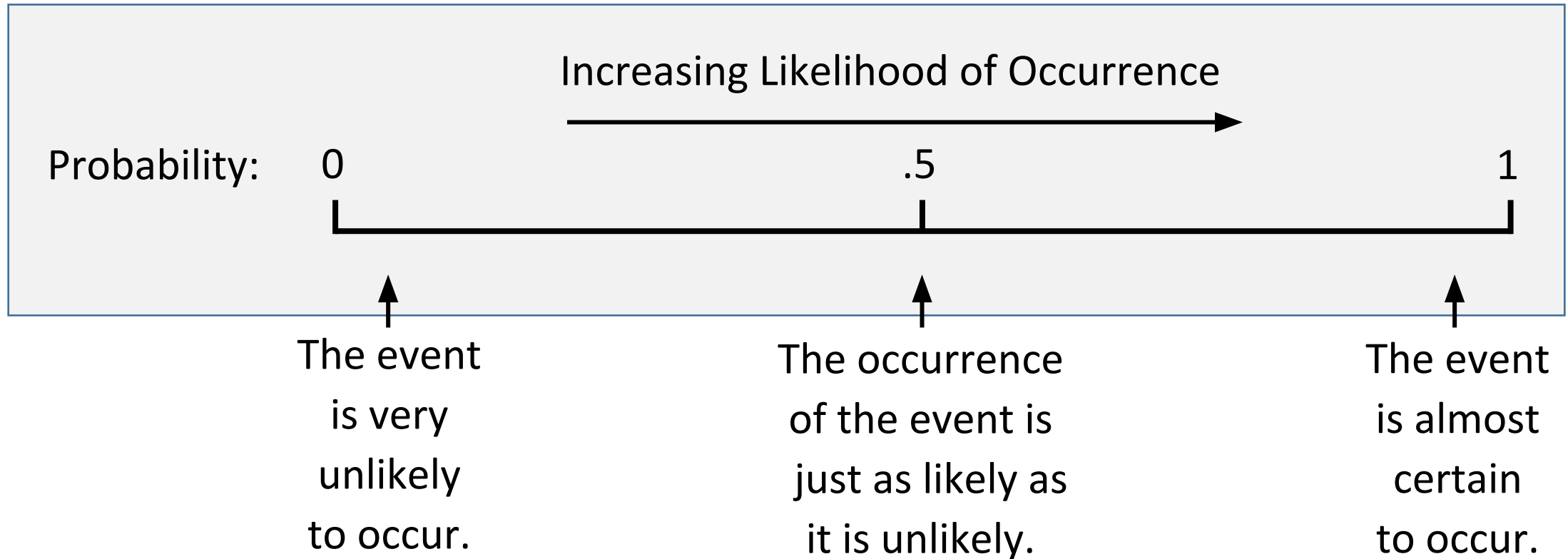
Uncertainties

- Managers often base their decisions on an analysis of uncertainties such as the following:
 - What are the *chances* that sales will decrease if we increase prices?
 - What is the *likelihood* a new assembly method will increase productivity?
 - What are the *odds* that a new investment will be profitable?

Probability

- Probability is a numerical measure of the likelihood that an event will occur.
- Probability values are always assigned on a scale from 0 to 1.
- A probability near zero indicates an event is quite unlikely to occur.
- A probability near one indicates an event is almost certain to occur.

Probability as a Numerical Measure of the Likelihood of Occurrence



Statistical Experiments

- In statistics, the notion of an experiment differs somewhat from that of an experiment in the physical sciences.
- In statistical experiments, probability determines outcomes.
- Even though the experiment is repeated in exactly the same way, an entirely different outcome may occur.
- For this reason, statistical experiments are sometimes called *random experiments*.

Random Experiment and Its Sample Space

- A Random experiment is a process that generates well-defined experimental outcomes.
- The sample space for an experiment is the set of all experimental outcomes.
- An experimental outcome is also called a sample point.

Random Experiment and Its Sample Space

Experiment

Toss a coin

Inspect a part

Conduct a sales call

Roll a die

Play a football game

Experiment Outcomes

Head, tail

Defective, non-defective

Purchase, no purchase

1, 2, 3, 4, 5, 6

Win, lose, tie

Random Experiment and Its Sample Space

- Example: Bradley Investments

Bradley has invested in two stocks, Markley Oil and Collins Mining. Bradley has determined that the possible outcomes of these investments three months from now are as follows:

Investment Gain or Loss in 3 Months (in \$1000s)	
<u>Markley Oil</u>	<u>Collins Mining</u>
10	8
5	-2
0	
-20	

A Counting Rule for Multiple-Step Experiments

- If an experiment consists of a sequence of k steps in which there are n_1 possible results for the first step, n_2 possible results for the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2) \dots (n_k)$.
- A helpful graphical representation of a multiple-step experiment is a tree diagram.

A Counting Rule for Multiple-Step Experiments

- Example: Bradley Investments
 - Bradley Investments can be viewed as a two-step experiment. It involves two stocks, each with a set of experimental outcomes.

Markley Oil: $n_1 = 4$

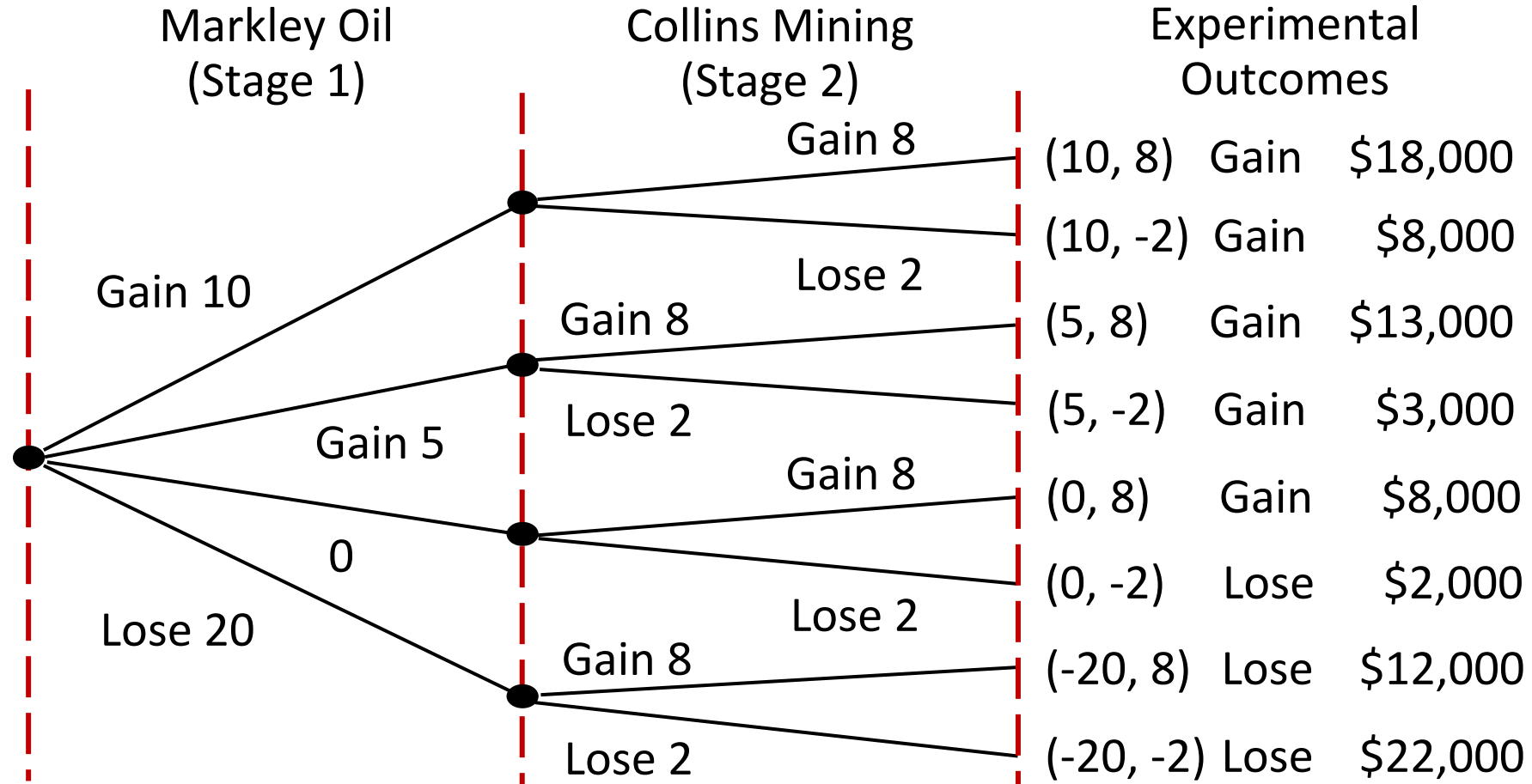
Collins Mining: $n_2 = 2$

Total Number of

Experimental Outcomes: $n_1 n_2 = (4)(2) = 8$

Tree Diagram

- Example: Bradley Investments



Counting Rule for Combinations

- Number of Combinations of N Objects Taken n at a Time
 - A second useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects.

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where: $N! = N(N - 1)(N - 2) \dots (2)(1)$

$$n! = n(n - 1)(n - 2) \dots (2)(1)$$

$$0! = 1$$

Counting Rule for Permutations

- Number of Permutations of N Objects Taken n at a Time
- A third useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects, where the order of selection is important.

$$P_n^N = n! \binom{N}{n} = \frac{N!}{(N-n)!}$$

where: $N! = N(N - 1)(N - 2) \dots (2)(1)$

$$n! = n(n - 1)(n - 2) \dots (2)(1)$$

$$0! = 1$$

Assigning Probabilities

- Basic Requirements for Assigning Probabilities
 1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively.

$$0 \leq P(E_i) \leq 1 \text{ for all } i$$

where: E_i is the i th experimental outcome
and $P(E_i)$ is its probability

Assigning Probabilities

- Basic Requirements for Assigning Probabilities
 2. The sum of the probabilities for all experimental outcomes must equal 1.

$$P(E_1) + P(E_2) + \dots + P(E_n) = 1$$

where: n is the number of experimental outcomes

Assigning Probabilities

- Classical Method

Assigning probabilities based on the assumption of equally likely outcomes

- Relative Frequency Method

Assigning probabilities based on experimentation or historical data

- Subjective Method

Assigning probabilities based on judgment

Classical Method

- Example: Rolling a Die

If an experiment has n possible outcomes, the classical method would assign a probability of $1/n$ to each outcome.

Experiment: Rolling a die

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

Probabilities: Each sample point has a $1/6$ chance of occurring

Relative Frequency Method

- Example: Lucas Tool Rental

Lucas Tool Rental would like to assign probabilities to the number of car polishers it rents each day. Office records show the following frequencies of daily rentals for the last 40 days.

<u>Number of Polishers Rented</u>	<u>Number of Days</u>
0	4
1	6
2	18
3	10
4	2

Relative Frequency Method

- Example: Lucas Tool Rental

Each probability assignment is given by dividing the frequency (number of days) by the total frequency (total number of days).

<u>Number of Polishers Rented</u>	<u>Number of Days</u>	<u>Probability</u>
0	4	$\frac{4}{40} = .10$
1	6	.15
2	18	.45
3	10	.25
4	<u>2</u>	<u>.05</u>
	40	1.00

Subjective Method

- When economic conditions or a company's circumstances change rapidly it might be inappropriate to assign probabilities based solely on historical data.
- We can use any data available as well as our experience and intuition, but ultimately a probability value should express our degree of belief that the experimental outcome will occur.
- The best probability estimates often are obtained by combining the estimates from the classical or relative frequency approach with the subjective estimate.

Subjective Method

- Example: Bradley Investments

An analyst made the following probability estimates.

<u>Experimental Outcome</u>	<u>Net Gain or Loss</u>	<u>Probability</u>
(10, 8)	\$18,000 Gain	.20
(10, -2)	\$8,000 Gain	.08
(5, 8)	\$13,000 Gain	.16
(5, -2)	\$3,000 Gain	.26
(0, 8)	\$8,000 Gain	.10
(0, -2)	\$2,000 Loss	.12
(-20, 8)	\$12,000 Loss	.02
(-20, -2)	\$22,000 Loss	.06
		1.00

Events and Their Probabilities

- An event is a collection of sample points.
- The probability of any event is equal to the sum of the probabilities of the sample points in the event.
- If we can identify all the sample points of an experiment and assign a probability to each, we can compute the probability of an event.

Events and Their Probabilities

- Example: Bradley Investments

Event M = Markley Oil Profitable

$$M = \{(10, 8), (10, -2), (5, 8), (5, -2)\}$$

$$P(M) = P(10, 8) + P(10, -2) + P(5, 8) + P(5, -2)$$

$$= .20 + .08 + .16 + .26$$

$$= .70$$

Events and Their Probabilities

- Example: Bradley Investments

Event C = Collins Mining Profitable

$$C = \{(10, 8), (5, 8), (0, 8), (-20, 8)\}$$

$$P(C) = P(10, 8) + P(5, 8) + P(0, 8) + P(-20, 8)$$

$$= .20 + .16 + .10 + .02$$

$$= .48$$

Some Basic Relationships of Probability

- There are some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities.

Complement of an Event

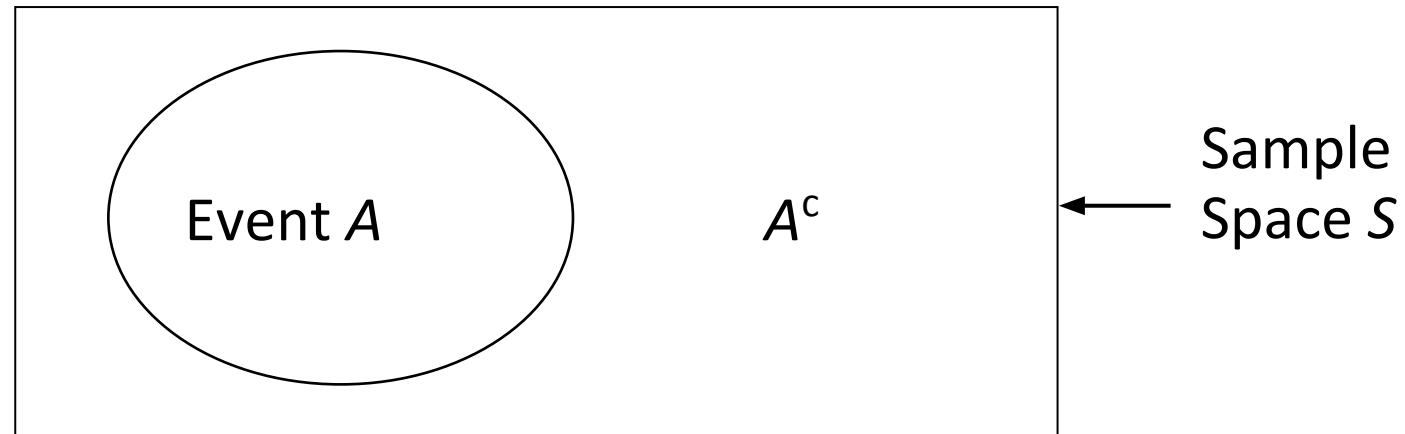
Union of Two Events

Intersection of Two Events

Mutually Exclusive Events

Complement of an Event

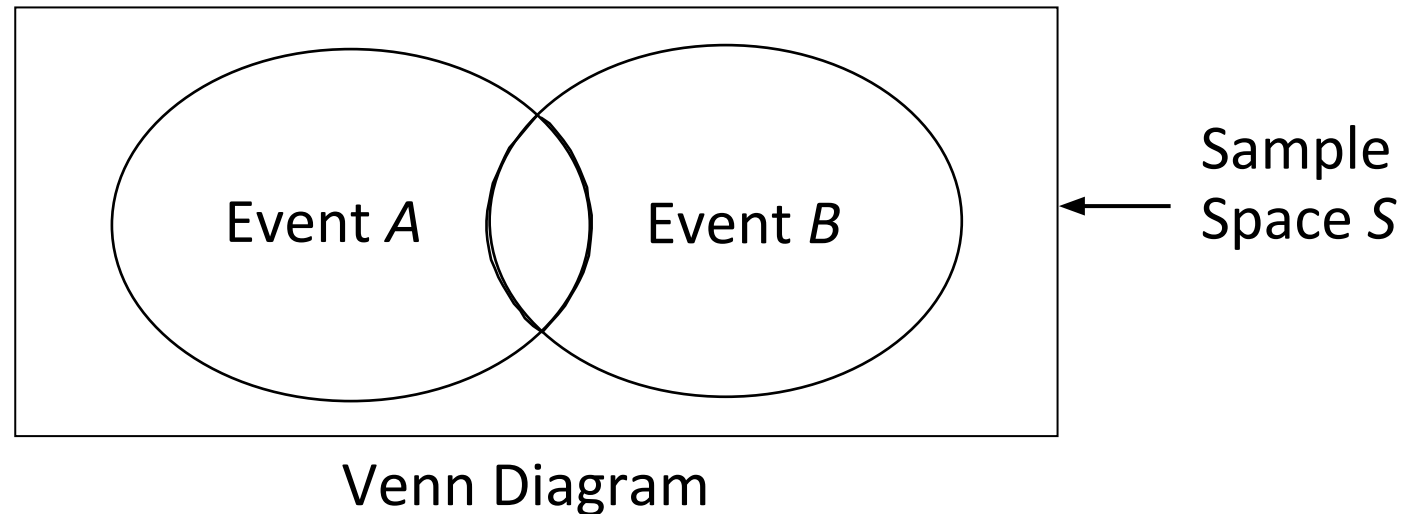
- The complement of event A is defined to be the event consisting of all sample points that are *not* in A .
- The complement of A is denoted by A^c .



Venn Diagram

Union of Two Events

- The union of events A and B is the event containing all sample points that are in A or B or both.
- The union of events A and B is denoted by $A \cup B$.



Union of Two Events

- Example: Bradley Investments

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

$M \cup C$ = Markley Oil Profitable

or Collins Mining Profitable (or both)

$$M \cup C = \{(10, 8), (10, -2), (5, 8), (5, -2), (0, 8), (-20, 8)\}$$

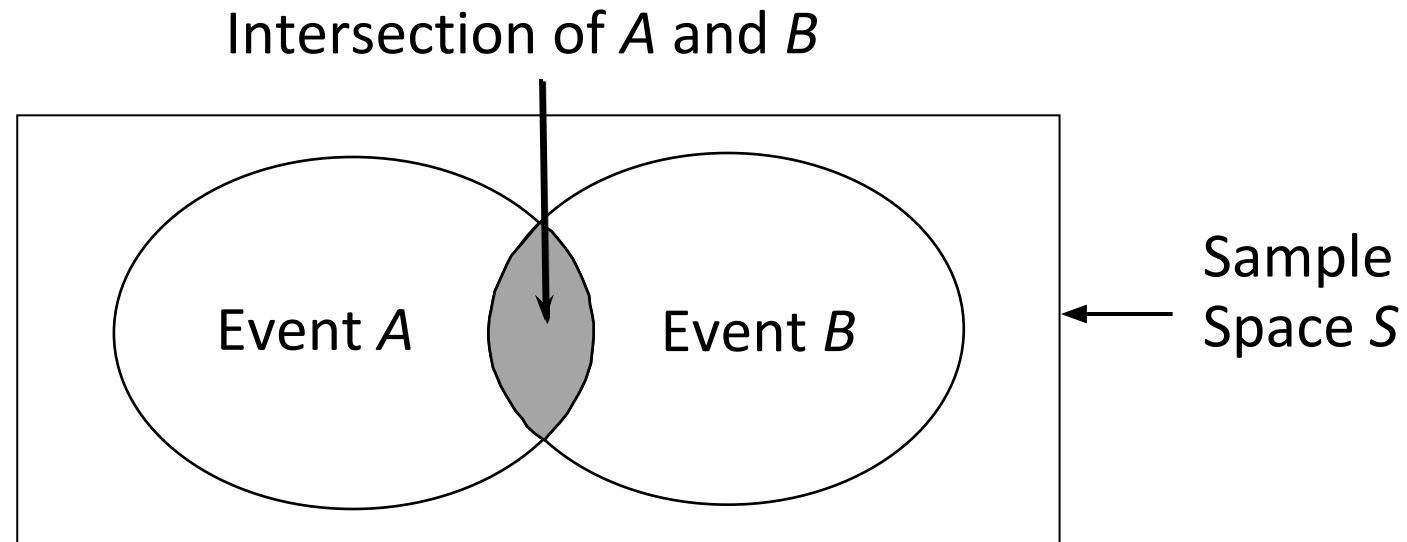
$$P(M \cup C) = P(10, 8) + P(10, -2) + P(5, 8) + P(5, -2) + P(0, 8) + P(-20, 8)$$

$$= .20 + .08 + .16 + .26 + .10 + .02$$

$$= .82$$

Intersection of Two Events

- The intersection of events A and B is the set of all sample points that are in both A and B .
- The intersection of events A and B is denoted by $A \cap B$.



Venn Diagram

Intersection of Two Events

- Example: Bradley Investments

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

$M \cap C$ = Markley Oil Profitable and Collins Mining Profitable

$$M \cap C = \{(10, 8), (5, 8)\}$$

$$P(M \cap C) = P(10, 8) + P(5, 8)$$

$$= .20 + .16$$

$$= .36$$

Addition Law

- The addition law provides a way to compute the probability of event A , or B , or both A and B occurring.
- The law is written as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Addition Law

- Example: Bradley Investments

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

$M \cup C$ = Markley Oil Profitable or Collins Mining Profitable

We know: $P(M) = .70$, $P(C) = .48$, $P(M \cap C) = .36$

Thus: $P(M \cup C) = P(M) + P(C) - P(M \cap C)$

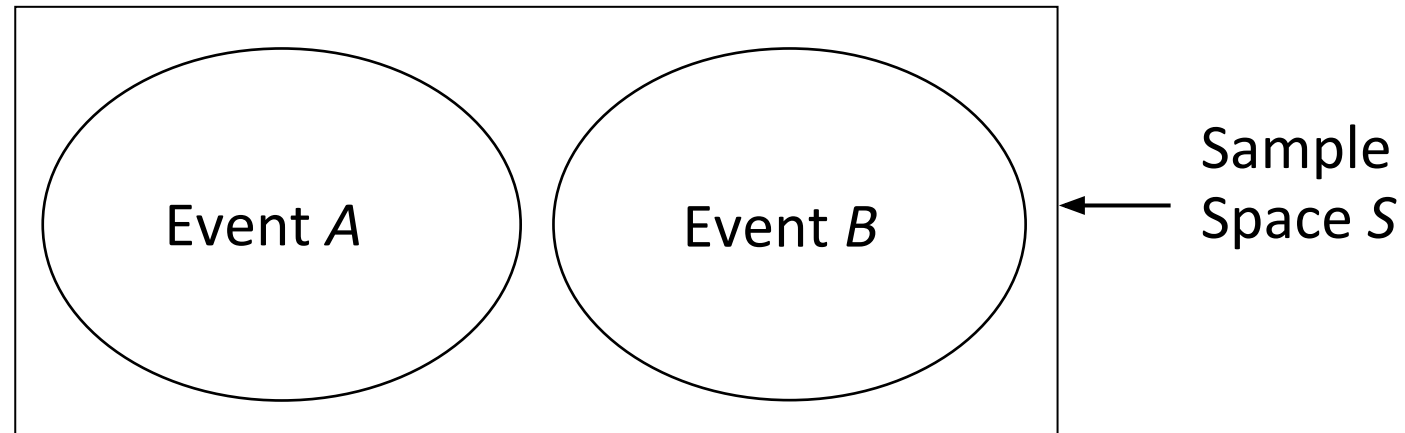
$$= .70 + .48 - .36$$

$$= .82$$

(This result is the same as that obtained earlier using the definition of the probability of an event.)

Mutually Exclusive Events

- Two events are said to be mutually exclusive if the events have no sample points in common.
- Two events are mutually exclusive if, when one event occurs, the other cannot occur.



Venn Diagram

Mutually Exclusive Events

- If events A and B are mutually exclusive, $P(A \cap B) = 0$.
- The addition law for mutually exclusive events is:

$$P(A \cup B) = P(A) + P(B)$$

Conditional Probability

- The probability of an event given that another event has occurred is called a conditional probability.
- The conditional probability of A given B has already occurred is denoted by $P(A|B)$.
- A conditional probability is computed as follows :

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

- Example: Bradley Investments

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

$P(C|M)$ = Collins Mining Profitable given Markley Oil Profitable

We know: $P(M \cap C) = .36$, $P(M) = .70$

$$\text{Thus: } P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{.36}{.70} = .5143$$

Multiplication Law

- The multiplication law provides a way to compute the probability of the intersection of two events.
- The law is written as:

$$P(A \cap B) = P(B)P(A | B)$$

or

$$P(A \cap B) = P(A)P(B | A)$$

Multiplication Law

- Example: Bradley Investments

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

$M \cap C$ = Markley Oil Profitable and Collins Mining Profitable

We know: $P(M) = .70$, $P(C|M) = .5143$

Thus: $P(M \cap C) = P(M)P(M/C)$

$$= (.70)(.5143)$$

$$= .36$$

(This result is the same as that obtained earlier using the definition of the probability of an event.)

Joint Probability Table

<u>Markley Oil</u>	<u>Collins Mining</u>		Total
	Profitable (C)	Not Profitable (C^c)	
Profitable (M)	.36	.34	.70
Not Profitable (M^c)	.12	.18	.30
Total	.48		
	.52		1.00

- Joint probabilities appear in the body of the table.
- Marginal probabilities appear in the margins of the table.

Independent Events

- If the probability of event A is not changed by the existence of event B , we would say that events A and B are independent.
- Two events A and B are independent if:

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Multiplication Law for Independent Events

- The multiplication law also can be used as a test to see if two events are independent.
- The law is written as:

$$P(A \cap B) = P(A)P(B)$$

Multiplication Law for Independent Events

- Example: Bradley Investments

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

Are events M and C independent?

Does $P(M \cap C) = P(M)P(C)$?

We know: $P(M \cap C) = .36$, $P(M) = .70$, $P(C) = .48$

But: $P(M)P(C) = (.70)(.48) = .34$, not $.36$

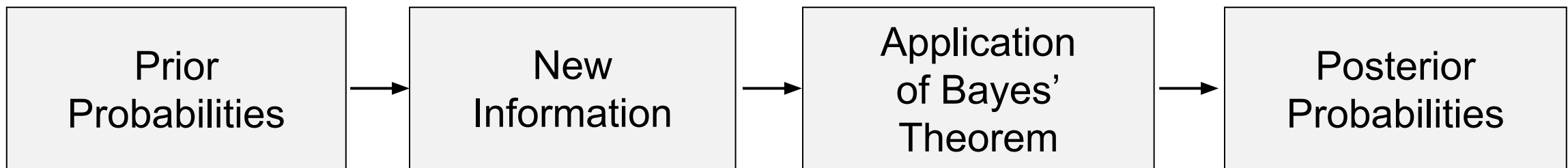
Hence: M and C are not independent.

Mutual Exclusiveness and Independence

- Do not confuse the notion of mutually exclusive events with that of independent events.
- Two events with nonzero probabilities cannot be both mutually exclusive and independent.
- If one mutually exclusive event is known to occur, the other cannot occur.; thus, the probability of the other event occurring is reduced to zero (and they are therefore dependent).
- Two events that are not mutually exclusive, might or might not be independent.

Bayes' Theorem

- Often we begin probability analysis with initial or prior probabilities.
- Then, from a sample, special report, or a product test we obtain some additional information.
- Given this information, we calculate revised or posterior probabilities.
- Bayes' theorem provides the means for revising the prior probabilities.



Bayes' Theorem

- Example: L. S. Clothiers

A proposed shopping center will provide strong competition for downtown businesses like L. S. Clothiers. If the shopping center is built, the owner of L. S. Clothiers feels it would be best to relocate to the shopping center.

The shopping center cannot be built unless a zoning change is approved by the town council. The planning board must first make a recommendation, for or against the zoning change, to the council.

Prior Probabilities

- Example: L. S. Clothiers

Let:

A_1 = town council approves the zoning change

A_2 = town council disapproves the change

Using subjective judgment:

$$P(A_1) = .7, \quad P(A_2) = .3$$

New Information

- Example: L. S. Clothiers

The planning board has recommended against the zoning change. Let B denote the event of a negative recommendation by the planning board.

Given that B has occurred, should L. S. Clothiers revise the probabilities that the town council will approve or disapprove the zoning change?

Conditional Probabilities

- Example: L. S. Clothiers

Past history with the planning board and the town council indicates the following:

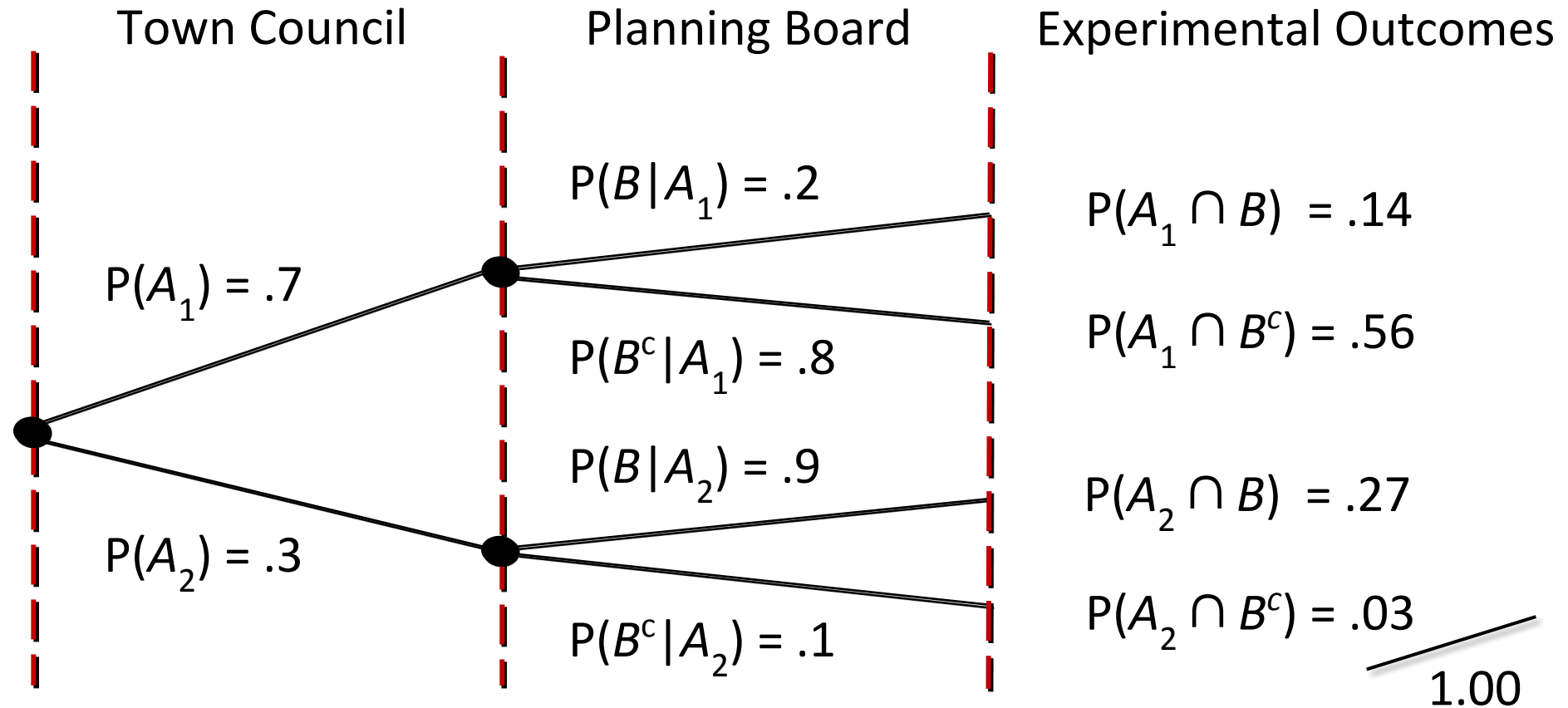
$$P(B|A_1) = .2 \quad \text{and} \quad P(B|A_2) = .9$$

Hence:

$$P(B^c|A_1) = .8 \quad \text{and} \quad P(B^c|A_2) = .1$$

Tree Diagram

- Example: L. S. Clothiers



Bayes' Theorem

- To find the posterior probability that event A_i will occur given that event B has occurred, we apply Bayes' theorem.

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}$$

- Bayes' theorem is applicable when the events for which we want to compute posterior probabilities are mutually exclusive and their union is the entire sample space.

Posterior Probabilities

- Example: L. S. Clothiers

Given the planning board's recommendation not to approve the zoning change, we revise the prior probabilities as follows:

$$\begin{aligned}P(A_1|B) &= \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} \\&= \frac{(.7)(.2)}{(.7).2 + (.3).9} \\&= .34\end{aligned}$$

Posterior Probabilities

- Example: L. S. Clothiers

The planning board's recommendation is good news for L. S. Clothiers. The posterior probability of the town council approving the zoning change is .34 compared to a prior probability of .70.

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers

- Step 1

Prepare the following three columns:

Column 1 - The mutually exclusive events for which posterior probabilities are desired.

Column 2 - The prior probabilities for the events.

Column 3 - The conditional probabilities of the new information *given* each event.

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers
 - Step 1

(1)	(2)	(3)	(4)	(5)
Events	Prior	Conditional		
A_i	Probabilities $P(A_i)$	Probabilities $P(B A_i)$		
A_1	.7	.2		
A_2	<u>.3</u>	.9		
	1.0			

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers

- Step 2

Prepare the fourth column

Column 4

Compute the joint probabilities for each event and the new information B by using the multiplication law.

Multiply the prior probabilities in column 2 by the corresponding conditional probabilities in column 3. That is, $P(A_i \cap B) = P(A_i) P(B|A_i)$.

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers
 - Step 2

(1)	(2)	(3)	(4)	(5)
Events	Prior	Conditional	Joint	
A_i	Probabilities $P(A_i)$	Probabilities $P(B A_i)$	Probabilities $P(A_i B)$	
A_1	.7	.2	.14 = .7(.2)	
A_2	<u>.3</u>	.9	<u>.27</u>	
	1.0			

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers

- Step 2 (continued)

We see that there is a .14 probability of the town council approving the zoning change and a negative recommendation by the planning board.

There is a .27 probability of the town council disapproving the zoning change and a negative recommendation by the planning board.

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers
 - Step 3

Sum the joint probabilities in Column 4. The sum is the probability of the new information, $P(B)$. The sum $.14 + .27$ shows an overall probability of .41 of a negative recommendation by the planning board.

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers
 - Step 3

(1)	(2)	(3)	(4)	(5)
Events	Prior	Conditional	Joint	
A_i	Probabilities $P(A_i)$	Probabilities $P(B A_i)$	Probabilities $P(A_i B)$	
A_1	.7	.2	.14	
A_2	<u>.3</u>	.9	<u>.27</u>	
	1.0			$P(B) = .41$

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers

- Step 4

Prepare the fifth column:

Column 5

Compute the posterior probabilities using the basic relationship of conditional probability.

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

The joint probabilities $P(A_i \cap B)$ are in column 4 and the probability $P(B)$ is the sum of column 4.

Bayes' Theorem: Tabular Approach

- Example: L. S. Clothiers
 - Step 4

(1) Events A_i	(2) Prior Probabilities $P(A_i)$	(3) Conditional Probabilities $P(B A_i)$	(4) Joint Probabilities $P(A_i B)$	(5) Posterior Probabilities $P(A_i B)$
A_1	.7	.2	.14	.3415 = .14/.41
A_2	<u>.3</u>	.9	<u>.27</u>	<u>.6585</u>
	1.0		$P(B) = .41$	1.0000

End of Chapter 4

