Solution: Quiz 3

1. Prove or disprove the following statements.

   (a) For all integers a and b, if a is even and b is odd, then $a^2-(b+1)^2$ is even.

   (b) For all integers a and b, if $a^2-(b+1)^2$ is even, then a is even and b is odd.

   <u>**Solution:**</u>

(a) We can use the **direct proof** to show that this statement is true.

Suppose a is even and b is odd.
That is, $a=2n$ for some integer n.
and $b=2m+1$ We want to show that $a^2-(b+1)^2$ is even. In particular, by substitution,

$$\begin{aligned} a^2-(b+1)^2 &= (2n)^2-(2m+1+1)^2 \\ &= 4n^2-4(m^2+2m+1) \\ &= 4(n^2-m^2-2m-1)=2K, \qquad \text{where } \ K=2(n^2-m^2-2m-1). \end{aligned}$$

Note that, since n and m are integers, then n^2, m^2, $2m$ are also integers. That is, K has to be an integer. Therefore, by definition, $a^2-(b+1)^2=2K$, $K\in\mathbb{Z}$, is even. ■

(b) This statement is false. We can **disprove** this by showing that its negation is true. The negation of this statement is given by
" there exist integers a and b such that $a^2-(b+1)^2$ is even, but a is odd (not even) or b is even (not odd). "

Since the negation is in the form of existential statement, we can prove this by finding an odd number a and even number b such that $a^2-(b+1)^2$ is even.

Using the odd number $a=1$ and even number $b=2$ gives

$$a^2-(b+1)^2=1^2-(2+1)^2=1-9=-8$$

which is even, since $-8=2k$ where $k=-4\in\mathbb{Z}$. That is, we found a counterexample that makes the negation true and therefore the given statement is false. ■

<u>**Remark for (b):**</u> It is possible to use other values of a and b as counterexamples.
In fact, we can set a to be any odd number of the form $a=2n+1$, for some integer n, and set b to be any even number of the form $b=2m$, for some integer m. As a result,

$$a^2-(b+1)^2=(2n+1)^2-(2m+1)^2=4n^2+2n+1-(4m^2+2m+1)=2c,$$

is an even number, where $c:=2n^2+n-2m^2-m$ is an integer (using $m,n\in\mathbb{Z}$). Hence the negation is true and the original statement is false. ■