Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

$$2, 3, 4, 6$$

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2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\frac{p_s}{\pi_s}$$

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E[mX_a] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

$$p_s = m_s \pi_s$$

$$m_s = \delta \frac{u'(C_s^*)}{u'(C_0^*)} = \delta \frac{C_0^*}{C_s^*}$$

$$\therefore \quad p_s = \left(\delta \frac{C_0^*}{C_s^*} \right) \pi_s \quad \#$$

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3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E\left[m_{01} R_s\right]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

$$m_{01} = \frac{\delta u'(C_1)}{u'(C_0)} = \delta \left(\frac{C_1}{C_0}\right)^{\gamma - 1}$$

$$\text{FOC}: \quad \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} R_s = 1$$

$$\pi_s \delta (\gamma - 1) \left(\frac{C_s}{C_0}\right)^{\gamma-2} R_s \cdot d\left(\frac{C_s}{C_0}\right) + \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s = 0$$

$$\text{Rearrange}: \quad \frac{\cancel{\pi_s \delta} (\gamma-1) \left(\frac{C_s}{C_0}\right)^{\cancel{\gamma-2}\ -1} R_s \cdot d\left(\frac{C_s}{C_0}\right)}{\cancel{\pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1}}} = -dR_s$$

$$(\gamma - 1)\left(\frac{C_s}{C_0}\right)^{-1} R_s \cdot d\left(\frac{C_s}{C_0}\right) = -dR_s$$

$$\frac{R_s}{(C_s/C_0)} \cdot d\left(\frac{C_s}{C_0}\right) = -\frac{dR_s}{(\gamma - 1)}$$

$$\frac{R_s}{(C_s/C_0)} \cdot \frac{d(C_s/C_0)}{dR_s} = -\frac{1}{(\gamma-1)}$$

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4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

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$p = \frac{1}{R_f}$

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

[16] I thank Michael Cliff of Virginia Tech for suggesting this example.

b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

a) $p = \begin{bmatrix} \frac{1}{1.05} \\ 6 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix} \rightarrow X' = \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}$

find $p_1 = p'X^{-1}e_1 = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.2476$ #

$p_2 = p'X^{-1}e_2 = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.7048$ #

find risk-neutral prob.

$\hat{\pi}_1 = p_1 R_f = 0.2476(1.05) = 0.26$ #

$\hat{\pi}_2 = p_2 R_f = 0.7048(1.05) = 0.74$ #

b) $m_1 = \frac{p_1}{\pi_1} = \frac{0.2476}{0.5} = 0.4952$ #

$m_2 = \frac{p_2}{\pi_2} = \frac{0.7048}{0.5} = 1.4096$ #

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6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

a)

$$\begin{aligned}
0 &= E[m(R_i - R_f)] \\
&= E[(a + bR_m)(R_i - R_f)] \\
&= aE[R_i] - aR_f + bE[R_m R_i] - bR_f E[R_m] \\
&= a(E[R_i] - R_f) + b(E[R_m]E[R_i] + \mathrm{cov}(R_m, R_i) - R_f E[R_m]) \\
&= (E[R_i] - R_f)(a + bE[R_m]) + b\,\mathrm{cov}(R_m, R_i)
\end{aligned}$$

$$\begin{aligned}
E[R_i] - R_f &= \frac{-b\,\mathrm{cov}(R_m, R_i)}{a + bE[R_m]} \\
&= -\frac{\mathrm{cov}(R_m, R_i)}{\sigma_m^2} \cdot \frac{b\sigma_m^2}{a + bE[R_m]} \\
&= -\beta_i \left(\frac{b\sigma_m^2}{a + bE[R_m]} \right)
\end{aligned}$$

find

$$\begin{aligned}
\gamma &= \frac{E[R_i] - R_f}{\beta_i} \\
&= \frac{\left[-\cancel{\beta_i} \left(\frac{b\sigma_m^2}{a + bE[R_m]} \right) \right]}{\cancel{\beta_i}}
\end{aligned}$$

$$\therefore \quad \gamma = \frac{-b\sigma_m^2}{a + bE[R_m]}$$

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b) Risk-free asset : $\dfrac{1}{R_f} = E[a + bR_m]$

$$a = \frac{1}{R_f} - bE[R_m]$$

Market portfolio : $1 = E[(a + bR_m)R_m]$

$$= aE[R_m] + bE[R_m^2]$$

$$1 = aE[R_m] + b(\sigma_m^2 + E[R_m^2])$$

Replace a : $1 = \left(\dfrac{1}{R_f} - bE[R_m]\right)E[R_m] + b(\sigma_m^2 + E[R_m^2])$

$$= \frac{E[R_m]}{R_f} - \cancel{bE[R_m^2]} + b\sigma_m^2 + \cancel{bE[R_m^2]}$$

$$1 = \frac{E[R_m]}{R_f} + b\sigma_m^2$$

find b : $b\sigma_m^2 = 1 - \dfrac{E[R_m]}{R_f}$; $1 \times \dfrac{R_f}{R_f}$

$$\therefore \quad b = \frac{R_f - E[R_m]}{R_f \sigma_m^2} \quad \#$$

find a : $a = \dfrac{1}{R_f} - bE[R_m]$

$$= \frac{1}{R_f} - \left(\frac{R_f - E[R_m]}{R_f \sigma_m^2}\right)E[R_m] \quad ; \quad \frac{1}{R_f} \times \frac{\sigma_m^2}{\sigma_m^2}$$

$$= \frac{\sigma_m^2}{R_f \sigma_m^2} - \frac{R_f E[R_m] + E[R_m^2]}{R_f \sigma_m^2}$$

$$\therefore \quad a = \frac{\sigma_m^2 - R_f E[R_m] + E[R_m^2]}{R_f \sigma_m^2} \quad \#$$

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$$a + bE[R_m] = \frac{\sigma_m^2 - \cancel{R_f E[R_m]} + \cancel{E[R_m^2]}}{R_f \sigma_m^2} + \frac{\cancel{R_f E[R_m]} - \cancel{E[R_m^2]}}{R_f \sigma_m^2}$$

$$= \frac{\cancel{\sigma_m^2}}{R_f \cancel{\sigma_m^2}}$$

$$= \frac{1}{R_f}$$

show
$$\gamma = \frac{-b\sigma_m^2}{a + bE[R_m]} \quad = \frac{1}{R_f}$$

$$= -\left(\frac{R_f - E[R_m]}{\cancel{R_f \sigma_m^2}}\right) \cancel{\sigma_m^2 R_f}$$

$$\therefore \quad \gamma = E[R_m] - R_f$$

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