Cointegration Test for Long-run Relationship

Spurious Problem

If y_t and x_t are nonstationary series

OLS estimated result of model

$$y_t = \alpha + \beta x_t + u_t$$

can lead to spurious problem.

Spurious regression has significant t-statistics, but the results are without any theoretical meaning.

R^2 is higher than Durbin-Watson.

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Cointegration Test for Long-run Relationship

Cointegrated Time-series

If y_t and x_t are nonstationary, but their linear combination $u_t = y_t - \alpha - \beta x_t$ is stationary.

Then, y_t and x_t are cointegrated time-series.

Cointegration Test

Statistical test that tests whether the series are cointegrated or not.

Vector Error Correction Mechanism (VECM)

From VARs unrestricted model:

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix},$$

$$\begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix} \sim IID\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right)$$

Model in matrix form:

$$Y_t = B_0 + B_1 Y_{t-1} + \in_t$$

Vector Error Correction Mechanism (VECM)

From: $Y_t = B_0 + B_1 Y_{t-1} + \in_t$

Let eigenvalues and eigenvectors of B_1 be

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \vdots & \vdots \\ c_1 & c_2 \\ \vdots & \vdots \end{bmatrix}$$

Provided eigenvalues are distinct, the eigenvectors will be linear independent and C is nonsingular. Then,

$$C^{-1} B_1 C = \Lambda \quad \text{and} \quad B_1 = C \Lambda C^{-1}$$

Example:

Define new vector of variables (linear combination of x_t and y_t) S_t as

$$S_t = C^{-1} Y_t \quad \text{or} \quad X_t = C S_t$$

$$C^{-1} Y_t = C^{-1} B_0 + C^{-1} C \Lambda C^{-1} Y_{t-1} + C^{-1} \in_t$$

Then, $S_t = B_0^* + \Lambda S_{t-1} + \eta_t$

where: $B_0^* = C^{-1} B_0$ and $\eta_t = C^{-1} \in_t$

Thus,

$$s_{1t} = b_{10}^* + \lambda_1 s_{1t-1} + \eta_{1t}$$

$$s_{2t} = b_{20}^* + \lambda_2 s_{2t-1} + \eta_{2t}$$

Example:

From $$s_{1t} = b_{10}^{*} + \lambda_1 s_{1t-1} + \eta_{1t}$$
$$s_{2t} = b_{20}^{*} + \lambda_2 s_{2t-1} + \eta_{2t}$$

Case 1: $|\lambda_1| < 1$ and $|\lambda_2| < 1$

The two series y_t and x_t are stationary or $I(0)$.

Case 2: $\lambda_1 = 1$ and $|\lambda_2| < 1$

Series y_t is $I(0)$ and x_t is $I(1)$.

Case 3: $\lambda_1 = \lambda_2 = 1$

Series y_t and x_t are nonstationary.

Multivariate Cointegration Test

VAR(1) with two variables

$$Y_t = B_0 + B_1 Y_{t-1} + \in_t$$

Assume that the two series are both *I(1)*, the model can be written in form of VECM

$$\Delta Y_t = B_0 + \Pi Y_{t-1} + \in_t, \; \Pi = B_1 - I$$

where: $$\Pi = \begin{pmatrix} \alpha_1 & -\beta\alpha_1 \\ \alpha_2 & -\beta\alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} 1 & -\beta \end{pmatrix}$$

Then, VECM becomes:

$$\Delta y_t = b_{10} + \alpha_1 (y_{t-1} - \beta x_{t-1}) + e_{1t},$$
$$\Delta x_t = b_{20} + \alpha_2 (y_{t-1} - \beta x_{t-1}) + e_{2t}$$

Multivariate Cointegration Test

Cointegration in VAR(1) depend on the rank of Π.

Cointegration in VAR(p) models for m variables.

$$\Delta Y_t = B_0 + \Pi Y_{t-1} + \Gamma_1 \Delta Y_{t-1} + \cdots + \Gamma_{p-1} \Delta Y_{t-p+1} + \in_t ,$$

$$t = p+1, \ldots, n$$

Cointegration depend on the rank of Π *(r)*.

If $r = m$, all variables are stationary

If $1 \leq r \leq m-1$, there exists cointegration relations.

Multivariate Cointegration Test

VECM can be estimated by ML.

The maximal value of the log-likelihood, for given rank r of the matrix Π, is equal to

$$\log(L_{\max}(r)) = c - \frac{n-p}{2}\sum_{j=1}^{r}\log(1-\hat{\lambda}_j)$$

<u>Eigenvalues test</u> is to check how many of the canonical correlations differ significant from zero. $r = (number\ of\ significant\ \hat{\lambda}_j)$

<u>Johansen trace test</u> is LR-test.

$$LR(r) = 2(\log(L_{\max}(m)) - \log(L_{\max}(r))) = -(n-p)\sum_{j=r+1}^{m}\log(1-\hat{\lambda}_j)$$

Multivariate Cointegration Test

<u>Test for number of cointegration relations</u>

1. Test H_0: $r=0$ against H_1: $r>=1$
2. Test H_0: $r=1$ against H_1:$r>=2$
3. Iteratively test H_0: $rank(\Pi)=r$ against H_1: $rank(\Pi)>=r+1$.

Multivariate Cointegration Test

Trends in Johansen VECM Framework

$$\Delta Y_t = \alpha(\beta Y_{t-1} + \mu + \rho t) + \sum_{i=1}^{p-1} \Gamma_i \Delta Y_{t-i} + \nu + \tau t + e_t$$

Place restrictions on the trend terms:

1. Unrestricted trend
2. Restricted trend, $\tau=0$
3. Unrestricted Constant, $\tau=0$ and $\rho=0$
4. Restricted constant, $\tau=0$, $\rho=0$, and $v=0$
5. No trend, $\tau=0$, $\rho=0$, $v=0$, and $\mu=0$