DERIVATIVES OF MORE-THAN-ONE INDEPENDENT VARIABLE FUNCTION

*much of this lecture presentation drawn from Aj.Pattra's and Aj.Kittichai's lecture note

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Contents

Roadmap after midterm

- Derivatives of more-than-one independent variable functions
- Optimization without constraint: More-than-one independent variable cases
- Optimization under equality constraint
- Integration and application

1 Introduction

- This lecture illustrates calculus tools used to deal with functions with more than one independent variables. Consider the following functions;

$$\begin{array}{ccll} Q_d & = & Q_d(P_1, P_2, Y) & \\ Q(A, L) & = & AK^{\alpha}L^{1-\alpha} & \\ U & = & U(c, \ell) & ;\text{where } w(24-\ell) = pc \end{array}$$

- **Partial differentiation** is used to find the impact of a change in one independent variable on the dependent variable, holding everything else constant, we need to use.
- **Total differential** is used to find the total change in the dependent variable as one or more independent variables change at the same time.
- **Total derivative** is used when independent variable are related to one another

2 Partial Differentiation

- Consider $y = f(x_1, x_2, x_3, ..., x_n)$ where $x_1, x_2, x_3, ..., x_n$ are independent of one another
- Suppose only x_1 changes by Δx_1, the corresponding change in y is Δy
- Difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x_1, x_2, x_3, ..., x_n) - f(x_1, x_2, x_3, ..., x_n)}{\Delta x_1}$$

- Partial derivative of y with respect to x;

$$\boxed{f_1 \equiv \frac{\partial y}{\partial x} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}}$$

- The process of taking partial derivatives is called "**partial differentiation**"

2.1 First-order partial derivative

- Technique of partial differentiation: allow one variable to vary while holding all other independent variables constant while.
- Example 1: $y = f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$. Find $f_1(1.3)$ and $f_2(1,3)$.

- Example 2 : $y = f(u,v) = (u+4)(3u+2v)$. Find $f_u(2,1)$ and $f_v(2,1)$.

2.1.1 Geometric Interpretation of Partial Derivatives

- Suppose $Q = Q(K, L)$
- $\dfrac{\partial Q}{\partial L} = MP_L$
- $\dfrac{\partial Q}{\partial K} = MP_K$

2.1.2 Gradient Vectors

- The gradient vector of the function f(x1, x2, ..., xn) is an n-vector of all the partial derivatives

$$\nabla f(x_1, x_2, ..., x_n) = (f_1, f_2, f_3, ...f_n)$$

where $f_i \equiv \dfrac{\partial y}{\partial x_i}$.

- Example: Find the gradient vector of the production function $Q = aK^b L^{1-b}$,where $a > 0$and $0 < b < 1$.

Application 1: Partial Market Equilibrium

$$\begin{array}{lll} \text{Demand} & Q = a - bP & (a, b > 0) \\ \text{Supply} & Q = -c + dP & (c, d > 0) \end{array}$$

Solution: $P^* = \frac{(a+c)}{(b+d)}$, $Q* = \frac{(ad-bc)}{(b+d)}$

- Comparative-static derivatives are the partial derivatives of P* and Q* with respect to parameters (a, b, c, and d):

$\dfrac{\partial P^*}{\partial a}, \dfrac{\partial P^*}{\partial b}, \dfrac{\partial P^*}{\partial c}, \dfrac{\partial P^*}{\partial ....},$
$\dfrac{\partial Q^*}{\partial a}, \dfrac{\partial Q^*}{\partial b}, \dfrac{\partial Q^*}{\partial c}, \dfrac{\partial Q^*}{\partial d},$
Partial derivatives of P* with respect to parameters a, b, c, and d are:

Partial derivatives of Q* with respect to parameters a, b, c, and d are:

Application 2: Elasticities

- Two variables

  If z = f(x, y), the partial elasticity of z w.r.t. x and y are:

  $\varepsilon_{zx} = \frac{\partial z/z}{\partial x/x} = \frac{\partial z}{\partial x}(\frac{x}{z})$ and $\varepsilon_{zy} = \frac{\partial z/z}{\partial y/y} = \frac{\partial z}{\partial y}(\frac{y}{z})$

  When all variables are positive, elasticities can be expressed as logarithmic derivatives:

  $\varepsilon_{zx} = \frac{\partial lnz}{\partial lnx}$ and $\varepsilon_{zy} = \frac{\partial lnz}{\partial lny}$

  - n variables

    If z = f(x1, x2, ..., xn) = f(x), the elasticity of z w.r.t. xi when all other variables are held constant is:

    $\varepsilon_{zi} = \frac{\partial f(x)/f(x)}{\partial x_i/x_i} = \frac{\partial lnz}{\partial ln_{x_i}}$

Example: Given the demand function $Q_1 = a - bP_q cP_2 + mY$

where Y = income, P2 = the price of a substitute good.

- Own price elasticity of demand:

- Income elasticity of demand:

- Cross-price elasticity of demand:

Example : Given $Q1 = 100–P1 + 0.75P2 - 0.25P3 + 0.0075Y$

at Y= 10,000 , P_1 =10 , P_2 =20 , P_3 =40 and Q_1 =100 , find the different cross-price elasticities of demand.

- $\varepsilon_{12} =$

- $\varepsilon_{13} =$

- Given a linearly homogenous Cobb-Douglas production function

  $Q = F(K, L) = AK^{\alpha}L^{\beta}$

- The output elasticity of capital:

  $\varepsilon_{QK} =$

- The output elasticity of labor:

  $\varepsilon_{QL} =$

Application 3: Production Function

- Example: Given a production function

  $Q = 36KL – 2K^2 – 3L^2$

- Marginal product of capital is: $MP_k = 36L - 4K$
- Marginal product of labor is: $MP_L = 36K – 6L$
- If the marginal revenue (MR) at $K = 2$ and $L = 3$ is \$5, the marginal revenue product (MRP) for the third unit of L is:

  $MRP_{L=3} = MRxMP_L$

Application 4: Multipliers in Macroeconomics Models

- Consider a national-income model

$$
\begin{aligned}
&Y = C + I_0 + G_0 \\
&C = \alpha + \beta(Y – T) \qquad && (\alpha > 0; 0 < \beta < 1) \\
&T = \gamma + \delta Y \qquad && (\gamma > 0; 0 < \delta < 1)
\end{aligned}
$$

- Equilibrium national income (in reduced form):

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$$

- Comparative-static derivatives:

$$\frac{\partial Y^*}{\partial \alpha}; \frac{\partial Y^*}{\partial \beta}; \frac{\partial Y^*}{\partial I_0}; \frac{\partial Y^*}{\partial G_0}; \frac{\partial Y^*}{\partial \gamma}; \frac{\partial Y^*}{\partial \delta}$$

- Comparative-static derivatives with special policy significance:
- Government-expenditure multiplier:

$\frac{\partial Y^*}{\partial G_0}$

- Nonincome-tax multiplier:

$\frac{\partial Y^*}{\partial \gamma}$

- Partial derivatives of Y* w.r.t. the income tax rate (δ):

$\frac{\partial Y^*}{\partial \delta}$

2.2 Second -order partial derivative

- Consider the function $z = f(x, y)$, which give rise to:

  $f_x \equiv \frac{\partial z}{\partial x}$ and $f_y \equiv \frac{\partial z}{\partial y}$

- Since f_x is a function of x(and y) , we can determine the rate of change of f_x with respect to x, while y is fixed, by a secondorder partial derivative with respect to x:

  $f_{xx} \equiv \frac{\partial}{\partial x}(f_x)$ or $\frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$

- Similarly, the second-order partial derivative with respect to y is:

  $f_{yy} \equiv \frac{\partial}{\partial y}(f_y)$ or $\frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}(\frac{\partial z}{\partial y})$

- Also, since fx is a function of y and fy is a function of x, cross (or mixed) partial derivatives can be written as:

  $f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$and $f_{yz} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}(\frac{\partial z}{\partial x})$

- Young's Theorem:
  Let $y = f(x_1, x_2, \ldots, x_n)$ is twice continuously differentiable (C^2). Then,
  $\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial i}$; i $\neq$ j

- Let $y = f(x_1, x_2)$. The second-order partial derivatives can be written in a matrix form called "Hessian matrix" :

$$H = \begin{bmatrix} f_{11} f_{12} \\ f_{21} f_{22} \end{bmatrix}$$

- **Example**: Let $z = x^2 e^{-y}$ Find f_x, f_y and the Hessian matrix.

- Consider an agricultural production function
  $Q = F(K, L, T) = AK^{\alpha}L^{\beta}T^{\gamma} \qquad A > 0; \alpha > 0, \beta > 0, \gamma > 0)$
  Where K = capital, L = labor, and T = land.
  Marginal product of capital is: F_k=

  Marginal product of Labor is: F_L=

  Marginal product of Land is: F_T=

  Second-order partial derivatives:

  Cross partial derivatives:

3 Differentials&Total Differentials

3.1 Recall the definition of derivatives:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- $dy = f'(x)dx$
- dy = differential of y
- dx = differential of x

Example: Let $y = 3x^2 + 7x – 5$. Find dy.

- The process of finding the differential dy from a function $y = f(x)$ is called differentiation. (Note: for finding dy/dx, this is called differentiation with respect to x).
- Total differential of a function Y is the sum of the approximated changes from all parameters.
- The process of finding total differentials is called "total differentiation."
- Let $y = f(x_1, x_2)$
- $dy = f_1 dx_1 + f_2 dx_2$, where $f_1 = \frac{\partial y}{\partial x_1}, f_2 = \frac{\partial y}{\partial x_2}$

3.1.1 Example: Saving Function

Let$z = 5x^2 + 6y + 7$. Find dz =?

- Consider a saving function $S = S(Y, r)$where Y = income, i = interest rate.

The total change in S is approximated by the total differential:

$dS = (\frac{\partial S}{\partial Y})dY + (\frac{\partial S}{\partial r})dr$ or dS=

For a given change in Y, the resulting change in S can be approximated by:

$dS = (\frac{\partial S}{\partial Y})dY$ where = $(\frac{\partial S}{\partial Y})$marginal propensity to save.

For a given change in $r(dr)$, the resulting change in S can be approximated by:

$dS = (\frac{\partial S}{\partial r})dr$

3.1.2 Example 2: Utility Function

- General case of n independent variables:
- Example: $U = U(x1, x2, \ldots, xn)$
- Total differential of U is:

  OR

- Example: Let $U = 100x^0.5y^{0.5}$. Find $dU=$?
- $dU=$

3.2 Rule of Differntials

Rule 1	$dk = 0$
Rule 2	$d(cu^{n)} = cnu^{n-1}du$
Rule 3	$d(u \pm v) = du \pm dv$
Rule 4	$d(u \cdot v) = v \cdot du + u \cdot dv$
Rule 5	$d(\frac{u}{v}) = \left[\frac{v \cdot du + u \cdot dv}{v^2}\right]$
Rule 6	$d(u \pm v \pm w) = du \pm dv \pm dw$
Rule 7	$d(uvw) = vw \cdot du + uw \cdot dv + uv \cdot dw$

- Example 1: $y = 3x_1^2 + x_1x_2^2$
- Example 2: $y = \frac{(x_1+x_2)}{2x_1^2}$
- Example 3: $y = 3x_1(2x_2 - 1)(x_3 + 5)$

3.2.1 Application: Utility Function

- consider a utility function

  $U = Ax^a y^b$

- Marginal utility of $x : MU_x = aAx^{a-1}y^b$
- Marginal utility of $y : MU_y = bAx^a y^{b-1}$
- Marginal rate of substitution (MRS) as the slope of the indifference curve:

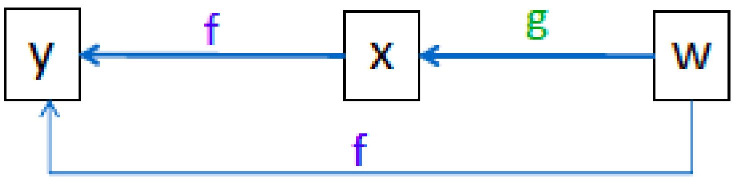
- Consider an agricultural production function

  $Q = F(K, L) = 60K^{0.25}L^{0.75}$

- Marginal product of capital is:
- Marginal product of labor is:
- Marginal rate of technical substitution (MRTS) as the slope of the isoquant:

4 Total Derivatives

- Now, consider the case where independent variables are related to one another.
- Example: what is the rate of change of $C(Y^*, T^0)$ w.r.t. T_0, where Y and T_0 are related?
- Consider $y = f(x, w)$where $x = g(w)$



- From $y = f[g(w), w]$
- Find $\frac{dy}{dw}$ =?

> 1. Use Chain Rule:
>
> $\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w = \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}$
>
> 2. Totally differentiate:
>
> $dy = f_x \cdot dx + f_w \cdot dw$
>
> $\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w$

- $\frac{dy}{dw}$is called the "total derivative of y w.r.t. w".
- $\frac{\partial y}{\partial w}$ (partial derivative) is a component of the total derivative.
- Example 1: $y = f(x, w) = 3x – w2$, where$x = g(w) = 2w^2 + w + 4$

  Find $\frac{dy}{dw}$

- Example 2: $U = U[c, g(c)]$

  Find $\frac{dU}{dc}$

- Let $y = f(x_1, x_2, w)$ where $\begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$

  Find $\frac{dY}{dw}$

- Example: $Q = Q(K, L, t)$, where $K = K(t)$ and $L = L(t)$

  Find $\frac{dQ}{dt}$

- $y = f(x1, x2, u, v)$where $\begin{cases} x_1 = g(u, v) \\ x_2 = h(u, v) \end{cases}$

- Suppose $U = U(C, n)$ where $C = consumption$, $n = leisure = 24 - L$, and $C = f(L)$.

5 Implicit function and its derivative

- Consider $y = f(x) = 3x^4$:explicit function
  - $y - 3x^4 = 0$:Implicit function
- General form: $F(y, x_1, x_2, \ldots, x_n) = 0$.
- This function may define an implicit function:

  $y = f(x_1, x_2, \ldots, x_n)$
- Given: $F(y, x_1, x_2, \ldots, x_n) = 0$
- We can write $dF = d(0) = 0$
- OR $F_y dy + F_1 dx_1 + F_2 dx_2 + \ldots + F_n dx_n = 0$ –(*)
- Implicit function has the total differential:
- $dy = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n$
- Substituting dy in (*) gives:

  $(F_y f_1 + F_1)dx_1 + (F_y f_2 + F_2)dx_2 + \ldots + (F_y f_n + F_n)dx_n = 0$

  $F_y f_i + F_i = 0$ for $i = 1, 2, \ldots, n$

  $\boxed{f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y} \text{ :Implicit Function Rule}}$

- Example 1: Find $\frac{dy}{dx}$ for the implicit function $y – 3x^4 = 0$.
- Example 2: Find $\partial y / \partial x$ for any implicit function(s) that may be defined by $F(y, x, w) = y^3 x^2 + w^3 + yxw – 3 = 0$
- Example 3: Assume $F(Q, K, L) = 0$ implicitly defines the production function $Q = f(K, L)$. Express MPP_k and MPP_L in relation to the function F.

5.1 Application 1: Partial Market Equilibrium

- Suppose now Q_d is a function of P as well as Y_0

$$
\begin{aligned}
Q_d &= Q_s \\
Q_d &= D(P, Y_0) \qquad \left(\tfrac{\partial D}{\partial P} < 0; \tfrac{\partial D}{\partial Y_0} > 0\right) \\
Q_s &= S(P) \qquad \left(\tfrac{\partial S}{\partial P} > 0\right)
\end{aligned}
$$

- Equilibrium condition:

  $D(P, Y^0) – S(P) = 0$

- Equilibrium price

  $P^* = P^*(Y_0)$

- Equilibrium identity

  $D(P^*, Y_0) – S(P^*) \equiv 0$,where $D(P^*, Y_0) – S(P^*) = F(P^*, Y_0)$

- Comparative-static derivatives:

  $\frac{dP^*}{dY_0}$

  $\frac{dQ^*}{dY_0}$

5.2 Application 2: Production Function

- Example : Given the equation for a production isoquant

  $F(K, L) = 16K^{0.25}L^{0.75} = 2144$

- Use the implicit function rule to find the slope of the isoquant
- HINT: ($|dK/dL|$: marginal rate of technical substitution