MA332 Linear Algebra

1. Find a 2×2 matrix A such that $\begin{bmatrix}4\\2\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ are eigenvectors of A, with eigenvalues -2 and 5 respectively. **(3 points)**

$A = PDP^{-1}$

$P = \begin{bmatrix}4 & 1\\ 2 & 1\end{bmatrix}$ $\therefore$ $P^{-1} = \begin{bmatrix}1/2 & -1/2\\ -1/2 & 2\end{bmatrix}$

$D = \begin{bmatrix}-2 & 0\\ 0 & 5\end{bmatrix}$

$A = \begin{bmatrix}-9 & 14\\ -7 & 12\end{bmatrix}$

2 (a) Use cofactor expansion method together with row operations to determine $|A|$ and $|A^{-1}|$ given

$$A = \begin{bmatrix}2 & -1 & 0 & 0\\ 1 & 2 & 3 & 1\\ 1 & 0 & 2 & 4\\ 1 & 0 & 2 & 5\end{bmatrix}.$$

$A = \begin{bmatrix}2 & -1 & 0 & 0\\ 1 & 2 & 3 & 1\\ 1 & 0 & 2 & 4\\ 1 & 0 & 2 & 5\end{bmatrix}$ $\therefore$ $|A| = 2\begin{vmatrix}2 & 3 & 1\\ 0 & 2 & 4\\ 0 & 2 & 5\end{vmatrix} + 1\begin{vmatrix}1 & 3 & 1\\ 1 & 2 & 4\\ 1 & 2 & 5\end{vmatrix}$

Since $\begin{vmatrix}1 & 3 & 1\\ 1 & 2 & 4\\ 1 & 2 & 5\end{vmatrix} \underset{\substack{R_2-R_1\\ R_3-R_1}}{=} \begin{vmatrix}1 & 3 & 1\\ 0 & -1 & 3\\ 0 & -1 & 4\end{vmatrix} \underset{R_3-R_2}{=} \begin{vmatrix}1 & 3 & 1\\ 0 & -1 & 3\\ 0 & 0 & 1\end{vmatrix} = (-1)$

$\therefore$ $|A| = (2)(2)\begin{vmatrix}2 & 4\\ 2 & 5\end{vmatrix} + (-1) = (2)(2)(10-8) - 1 = \boxed{7}$

$|A^{-1}| = \frac{1}{|A|} = \frac{1}{7}$.

(b) Let $B = \begin{bmatrix}3 & 1 & 2\\ 0 & 3 & 2\\ 0 & 0 & 9\end{bmatrix}$, determine det C *(det C is a positive value)* if $(\det B)^{\frac{1}{2}}\det(A^2A^T) = \dfrac{\det(C^2)}{\det(A^{-1})}$.

(5 points)

det B=81

det C = 21

3. Let $A = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}$

a) Find the eigenvalues of the singular matrix A.

b) Find a basis of R^3 consisting of eigenvectors of A.

c) By expressing (1,1,1) as a combination of eigenvectors or by diagonalizing $A = PDP^{-1}$, compute $A^{99}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(12 points)

a) $\lambda = 0, 1, 1$

b) $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $v_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 0$

v_2, v_3: Correspond to $\lambda = 1$ → 2 multiplicities

dim (eigenspace for 1) = 2 ✓

$2 + 1 = 3$ ✓ $= n$.

c) $A^{99}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$A^{99} = P\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{99} P^{-1} = PDP^{-1} = A$$

$$A^{99} = PD^{99}P^{-1} = A$$

$$A^{99}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$$