Practice problem set 7

Constrained optimization problem

Semester 1/2017

(Solution)

Question 1:

The profit obtained by a firm from producing and selling x and y units of two brands of a commodity is given by

$$P(x,y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$$

a. Find the production levels that maximize profits.

Ans. $Max_{x,y}\ P(x,y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$

$$\Rightarrow\ (x^*, y^*) = (230, 5)$$

b. A key raw material is rationed so that total production must be restricted to 200 units. Find the production levels that now maximize profits.

Ans. $Max_{x,y}\ P(x,y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$

Subject to $x + y = 200$

$$\Rightarrow\ (x^{**}, y^{**}) = (195, 5)$$

Question 2

Suppose that Mr. Bean's utility depends on two commodities: x_1 and x_2. His utility function is given by:

$$U = x_1x_2 + 2x_1 + x_2\ .$$

Suppose that Mr. Bean's income is \$76, and the per-unit prices for the commodities x_1 and x_2 are \$2 and \$3, respectively.

a. Determine the values x_1^* and x_2^* that maximize Mr. Bean's utility, given that Mr.Bean spends all of his income on these two commodities.

Ans. $Max_{x_1,x_2}\ U(x_1,x_2) = x_1x_2 + 2x_1 + x_2$

subject to $2x_1 + 3x_2 = 76$

$L(x_1,x_2,\lambda) = x_1x_2 + 2x_1 + x_2 + \lambda[76 - 2x_1 - 3x_2]$

FOC: $L_1 = 0 \Rightarrow x_2 + 2 - 2\lambda = 0$ -- (1)

$L_2 = 0 \Rightarrow x_1 + 1 - 3\lambda = 0$ -- (2)

$L_\lambda = 0 \Rightarrow 76 - 2x_1 - 3x_2 = 0$ -- (3)

$\Rightarrow x_1^* = 20,\ x_2^* = 20$

b. Show the second-order sufficient condition for the constrained utility maximization.

SOC: $L_{11} = 0;\ L_{21} = L_{12} = 1;\ L_{22} = 0$

$g_1 = 2;\ g_2 = 3$

$\Rightarrow$ Bordered Hessian:

$$|\bar{H}| = \begin{vmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} + (3)\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = (-2)(-3) + (3)(2-0) = 6 + 6 = 12 > 0.$$

Thus, U* = 720 is a maximum utility given the budget of \$76.

Question 3:

Suppose that a monopolistic firm sells a single product in three separate markets (say, three different countries), and therefore faces three different demand functions:

$$P_1 = 32 - 1.5Q_1$$
$$P_2 = 68 - 3Q_2$$
$$P_3 = 40 - 2Q_3$$

The total cost function is given by:

$$TC = 60 + 20Q \quad \text{where } Q = Q_1 + Q_2 + Q_3$$

a. Suppose that there is a shortage of a key raw material so that the total production must be restricted to $\overline{Q} = 12$ units. Find the profit-maximizing output levels of Q_1, Q_2, and Q_3, and determine the maximum profit.

Ans. $(Q_1^*, Q_2^*, Q_3^*) = (\frac{16}{9}, \frac{62}{9}, \frac{10}{3})$ and $\pi^* = 189.33$

b. Use the bordered Hessian matrix to verify that the second-order sufficient condition is met.

Ans.

$$|\overline{H}| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & -6 & 0 \\ 1 & 0 & 0 & -4 \end{vmatrix}$$

$|\overline{H_2}| = 9 > 0$ and $|\overline{H_3}| = -54 < 0$

Question 4

A firm produces and sells two commodities. By selling x tons of the first commodity the firm gets a price per ton given by $p = 96 - 4x$. By selling y tons of the other commodity the price per ton is given by $q = 84 - 2y$.

The cost of producing and selling x tons of the first commodity and y tons of the second is given by:

$$C(x, y) = 2x^2 + 2xy + y^2 \ .$$

a. Show that the firm's profit function is:

$$P(x, y) = -6x^2 - 3y^2 - 2xy + 96x + 84y$$

Ans.

$P(x, y) = (96 - 4x)x + (84 - 2y)y - (2x^2 + 2xy + y^2) = \ -6x^2 - 3y^2 - 2xy + 96x + 84y$

b. Compute the first-order partial derivatives of profit (P) with respect to x and y, and find its only stationary point.

Ans. $P_x = -12x - 2y + 96; \ P_y = -6y - 2x + 84$

$\Rightarrow \ (x^*, y^*) = (6, 12)$ and $P(x^*, y^*) = 792$

c. Suppose the production causes pollution, and that the authorities for this reason require the firm the produce only 11 tons in total. Solve the firm's maximization problem in this case. Verify that the production restrictions do reduce the maximum possible value of $P(x, y)$.

Ans. $(x^{**}, y^{**}) = (4, 7)$ and $P(x^{**}, y^{**}) = 673$

Question 5:

Given the utility maximization problem

$$Max_{x,y} U(x, y) = \sqrt{x} + y$$
$$\text{subject to} \quad x + 4y = 100$$

a. Find the quantities demanded of the two goods using the Lagrange method. Determine the maximum utility level and the value of the Lagrange multiplier.

Ans. $(x^*, y^*) = (4, 24)$ and $U(x^*, y^*) = 26$

$\lambda = 0.25$

b. Suppose income increases from 100 to 101. What is the exact increase in the optimal value of U(x, y)? Compare with the value found in (a) for the Lagrange multiplier.

Ans. $(x^{**}, y^{**}) = (4, 24.25) \Rightarrow U(x^{**}, y^{**}) = 26.25$

Thus, $U(x^{**}, y^{**}) - U(x^*, y^*) = 0.25 = \lambda$

c. Suppose we change the budget constraint to $px + qy = m$, but keep the same utility function. Derive the quantities demanded of the two goods if $m > \frac{q^2}{4p}$.

Ans. $x^* = \frac{q^2}{4p^2}$; $y^* = \frac{m}{q} - \frac{q}{4p}$

Here, $y^* > 0 \Leftrightarrow m > q^2/4p$

Question 6:

A firm has an order of 10,000 units of its product and has two plants at which to manufacture these units. Let q_1 be the number of units to be produced at the first plant and q_2 denote the number to be manufactured at the second plant. Suppose that the cost function is given by

$$C = 48q_1^3 + 3q_2^3 + 25,000.$$

a. Use the method of Lagrange multipliers to determine how many units should be produced at each plant to minimize this cost function.

$q_1 = 2,000$ *units* and $q_2 = 8,000$ *units*

b. Confirm your result by checking the second-order condition.

$$H = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 288q_1 & 0 \\ -1 & 0 & 18q_2 \end{bmatrix};$$

$|H| = -288q_1 - 18q_2 = -(288q_1 + 18q_2) < 0$.
This guarantees the solution obtained in "a" as a constrained minimizer.

Question 7: *Consumption-Leisure choice*

An individual has a Cobb–Douglas utility function $U(M, L) = AM^a L^b$, where m is income and l is leisure. Here A, a, and b are positive constants, with $a + b \leq 1$. A total of T_0 hours are allocated between work (W) and leisure (L), so that $W + L = T_0$. If the hourly wage is w, then $M = \overline{w}W$, and the individual's problem is

$$max_{M,L} \; AM^a L^b \; st. \; M/\overline{w} + L = T_0$$

a. Solve for optimal leisure that this individual will choose.

$M^* = \frac{a}{a+b}\overline{w}T_0$ and $L^* = \frac{b}{a+b}T_0$

b. Confirm your result by checking the second-order condition.

Check the determinant of the bordered Hessian matrix.

$$H = \begin{bmatrix} 0 & -1/\overline{w} & -1 \\ -1/\overline{w} & Aa(a-1)M^{a-2}L^b & AabM^{a-1}L^{b-1} \\ -1 & AabM^{a-1}L^{b-1} & Ab(b-1)M^aL^{b-2} \end{bmatrix};$$

$|H| = 2AabM^{a-1}L^{b-1}/\overline{w} - Aa(a-1)M^{a-2}L^b - Ab(b-1)M^aL^{b-2} > 0$.

(This follows from that $a-1<0$ and $b-1<0$. The two terms associated with a-1 and b-1 are both negative. Thus, sum of all the three terms must be greater than zero.)

This confirms the solution obtained in "a" as a constrained maximizer.

c. How does the change in hourly wage affect the decision to work?

$L^* = \frac{b}{a+b} T_0 \Rightarrow \partial L^* / \partial \overline{w} = 0 \Rightarrow$ nothing changes.

Question 8: *Intertemporal consumption problem*

Consider a consumer who lives in two periods of time. In the first period, the consumer earns income Y_0. This income can be allocated for consumption in the current period (C_0) or saving (S) for the future use. The amount of saving that the consumer chooses can be used in the subsequent period for consumption. Every unit of saving would yield this consumer $(1+r)$ of proceeds that can be used for the consumption in the future (C_1).

Let C_0 be the amount of today's consumption, C_1 be the amount of future's consumption, S be the amount of savings, and $1+r$ be gross yield from saving. The problem that this consumer faces is to choose for different level of consumption between the two periods, maximizing the following utility-based criteria function given by,

$U = ln(C_0) + \beta\, ln(C_1)$ subject to $C_1 = (1+r)(Y_0 - C_0)$
where $0 < \beta < 1$.

Consider the problem.

a. Solve for the optimal consumption in the two periods using the LaGrange multiplier method.

$C_0 = \frac{y_0}{1+\beta}$
$C_1 = \frac{\beta}{1+\beta}(1+r)y_0$

b. Confirm your result in a by checking the second-order condition.

$$H = \begin{bmatrix} 0 & -(1+r) & -1 \\ -(1+r) & -1/C_0^2 & 0 \\ -1 & 0 & -\beta/C_1^2 \end{bmatrix};$$

$$|H| = 1/C_0^2 + \beta(1+r)^2/C_1^2 > 0$$

c. What happen to C_0 if β increases. Explain the intuition of your result.
As β increases, the individual places more weight of future consumption. Thus, consumer is more willing to postpone more of today's resource for future consumption.

d. How does the interest rate affect the amount of saving? Explain the intuition of your result.

Not in this case. Saving remains the same. Theoretically, two forces determine the movement of saving rate. One is the substitution effect, and the other is income effect. To the former, an increase in interest rate induces consumer to cut their current consumption because of an increase in the price of today's consumption. To the latter, an increase in interest rate increase total resource of consumer, and thus causing an increase in current consumption. The two workforces counteract each other, and cancel out under this utility function, which is known under the name of iso-elasticity.

(Note: The intuition provided in this part can be applied to the analysis in question 8 above. In the last part of question 8, labor supply does not change with respect to the change in hourly wage. This occurs because substitution effect cancels out with the income effect.)

Question 9: *Cost minimization problem.*

Consider a representative firm with the production function taking the form: $Q = K^a + L^a$, where K is the capital, L is the labor input, Q is the amount of output, and $a \in (0,1)$. This firm purchases capital and labor for the market at the price per unit of r and w, respectively. Consider the following problem

a. Does the production function exhibit returns to scale technology? If yes, state the type of returns to scale.

Yes, the function is Homogenous of degree a. Since $a \in (0,1)$, the function exhibits a decreasing return to scale technology.

b. Use the second-order derivative matrix and show that the production function is concave.

H = [$a(a-1)K^{a-2}$ 0]
[0 $a(a-1)L^{a-2}$];

|H1| = $a(a-1)K^{a-2} < 0$

|H2| = $a^2(a-1)^2 K^{a-2} L^{a-2} > 0$

c. Derive the cost-minimizing bundle of factor inputs.

$$K = \frac{r^{\frac{1}{a-1}}}{\left[w^{\frac{a}{a-1}} + r^{\frac{a}{a-1}}\right]^{\frac{1}{a}}} Q^{\frac{1}{a}} \qquad L = \frac{w^{\frac{1}{a-1}}}{\left[w^{\frac{a}{a-1}} + r^{\frac{a}{a-1}}\right]^{\frac{1}{a}}} Q^{\frac{1}{a}}$$

d. Confirm your result in "c" that your proposed solution is the least cost combination of factor inputs.

H = [0 $-aK^{a-1}$ $-aL^{a-1}$]
[$-aK^{a-1}$ $-\lambda a(a-1)K^{a-2}$ 0]
[$-aL^{a-1}$ 0 $-\lambda a(a-1)L^{a-2}$];

$|H| = aL^{a-1} \cdot \lambda a(a-1)K^{a-2} \cdot aL^{a-1} + aK^{a-1} \cdot aK^{a-1} \cdot \lambda a(a-1)L^{a-2} < 0$ because $(a-1) < 0$.

e. Show the expression for LaGrange multiplier.

$$\lambda = \frac{1}{a}\frac{w^{\frac{a}{a-1}} + r^{\frac{a}{a-1}}}{\left[w^{\frac{a}{a-1}} + r^{\frac{a}{a-1}}\right]^{\frac{1}{a}}}Q^{\frac{1-a}{a}}$$

f. Derive the minimized cost function.

$$C(*) = \frac{w^{\frac{a}{a-1}} + r^{\frac{a}{a-1}}}{\left[w^{\frac{a}{a-1}} + r^{\frac{a}{a-1}}\right]^{\frac{1}{a}}}Q^{\frac{1}{a}}$$

g. Show that marginal cost is the LaGrange multiplier.

This can be evidently shown by comparing "e" with derivative of the expression obtained in "f" with respect to Q. They are both the same.

h. Show that the minimized cost function is concave in the prices of factor inputs. That is, show that the Hessian of minimized cost function is negative definite.

Question 10:

Solve for the following constrained optimization problem.

a. $min\ \ x^2 + y^2 + z^2\ \ st.\ \ x + y + z = 1$
   $(x, y, z) = (1/3, 1/3, 1/3)$

b. $max\ U(x,y,L) = 2xy(24-L)\ \ st.\ p_x x + p_y y\ = wL$

$(x,y,L) = (8\frac{w}{p_x}, 8\frac{w}{p_y}, 16)$