Examples for Assignment 5

1. Let $f(x) = x^2 + 5$ and $g(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{1}{x}, & 2 \le x < 3. \end{cases}$ Find $f \circ g$ and $g \circ f$.

2. Define a function $g : [0, 3) \to \mathbb{R}$ and $h : [0, 3) \to \mathbb{R}$ as

$$g(x) = \frac{1}{x+1}, \quad \text{and} \quad h = \frac{12}{x+1} - 5,$$

and define a function $f : [-1, \infty) \to [0, \infty)$ be a function

$$f(x) = \begin{cases} x + 1, & x \in [-1, 4) \\ \sqrt{x} + 3, & x \in [4, \infty). \end{cases}$$

(a) Find domains, co-domains, and ranges of f, g and h.

(b) Find the composite functions $f \circ g$, $g \circ f$ and $f \circ h$ together with their domains and ranges.

3. Define a function $f : [-1, \infty) \to [0, \infty)$ be a function $f(x) = \begin{cases} x + 1, & x \in [-1, 4) \\ \sqrt{x} + 3, & x \in [4, \infty). \end{cases}$

Show that f is injective and find its inverse function f^{-1}.

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Solution

1. Let $f(x) = x^2 + 5$ and $g(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{1}{x}, & 2 \le x < 3. \end{cases}$ Find $f \circ g$ and $g \circ f$.

<u>**Answer:**</u> For $f(x) = x^2 + 5$, the domain is $D_f = \mathbb{R}$ and the range is $R_f = [5, \infty)$ (because $x^2 \ge 0 \quad \Rightarrow \quad x^2 + 5 \ge 5$).

For $g(x)$, we see from the definition of $g(x)$ that the domain is $(-\infty, 2) \cup [2, 3) = (-\infty, 3)$.
(i) For $x \in (-\infty, 2)$, $g(x) = 3 - x$, and $g(x) \in (1, \infty)$ because

$$x \in (-\infty, 2) \quad \Rightarrow \quad -x > -2 \quad \Rightarrow \quad 3 - x > 3 - 2 \quad \Rightarrow \quad g(x) > 1.$$

(ii) For $x \in [2, 3)$, $g(x) = \frac{1}{x}$, and $g(x) \in (\frac{1}{3}, \frac{1}{2}]$ because

$$x \in [2, 3) \quad \Rightarrow \quad 2 \le x < 3 \quad \Rightarrow \quad \frac{1}{3} < \frac{1}{x} \le \frac{1}{2} \quad \Rightarrow \quad g(x) \in (\frac{1}{3}, \frac{1}{2}].$$

The range of g is $(\frac{1}{3}, \frac{1}{2}] \cup (1, \infty)$.

► <u>$f \circ g$</u>

Notice that $D_f \cap R_g = \mathbb{R} \cap \{(\frac{1}{3}, \frac{1}{2}] \cup (1, \infty)\} \ne \emptyset$, so we can construct $f \circ g$. Moreover, since $R_g \subset D_f$, the domain of $f \circ g$ is the same as $D_g = (-\infty, 3)$ and for $x \in (-\infty, 3)$,
if $x \in (-\infty, 2)$, $g(x) = 3 - x$ so that $(f \circ g)(x) = f(g(x)) = f(3 - x) = (3 - x)^2 + 5$, and
if $x \in [2, 3)$, $g(x) = \frac{1}{x}$ so that $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x}) = \frac{1}{x^2} + 5$

$$(f \circ g)(x) = \begin{cases} (3 - x)^2 + 5, & x < 2 \\ \frac{1}{x^2} + 5, & 2 \le x < 3. \end{cases}$$

► <u>$g \circ f$</u>
Notice that $D_g \cap R_f = (-\infty, 3) \cap [5, \infty) = \emptyset$, so we **cannot** construct $g \circ f$. ■

2. Define a function $g : [0, 3) \to \mathbb{R}$ and $h : [0, 3) \to \mathbb{R}$ as

$$g(x) = \frac{1}{x+1}, \quad \text{and} \quad h = \frac{12}{x+1} - 5,$$

and define a function $f : [-1, \infty) \to [0, \infty)$ be a function

$$f(x) = \begin{cases} x+1, & x \in [-1, 4) \\ \sqrt{x}+3, & x \in [4, \infty). \end{cases}$$

(a) Find domains, co-domains, and ranges of f and g.

(b) Find the composite functions $f \circ g$, $g \circ f$ and $f \circ h$ together with their domains and ranges.

Answer:

(a) ► Consider function $g : [0, 3) \to \mathbb{R}$, $g(x) = \frac{1}{x+1}$. Since g is defined from $[0, 3)$ to $\mathbb{R}$

- the domain is $[0, 3)$,
- the co-domain is $\mathbb{R}$.

To find the range of g, consider $x \in [0, 3)$, or $0 \le x < 3$,

$$\begin{array}{rcl} 0+1 \le & x+1 & < 3+1 \\ 1 \le x+1 & \text{and} & x+1 < 4 \\ \dfrac{1}{x+1} \le 1 & \text{and} & \dfrac{1}{4} < \dfrac{1}{x+1} \\ \dfrac{1}{4} < & \frac{1}{x+1} & \le 1 \\ \dfrac{1}{4} < & g(x) & \le 1. \end{array}$$

So the range of g is $(\frac{1}{4}, 1]$.

► Consider function $h : [0, 3) \to \mathbb{R}$, $h(x) = \frac{12}{x+1} - 5$. Since h is defined from $[0, 3)$ to $\mathbb{R}$

- the domain is $[0, 3)$,
- the co-domain is $\mathbb{R}$.

To find the range of h, consider $x \in [0, 3)$, or $0 \le x < 3$, similar to g, we have

$$\begin{array}{rcl} 0+1 \le & x+1 & < 3+1 \\ \dfrac{1}{4} < & \frac{1}{x+1} & \le 1 \\ \dfrac{12}{4} - 5 < & \frac{12}{x+1} - 5 & \le 12 - 5 \\ -2 < & h(x) & \le 7. \end{array}$$

So the range of h is $(-2, 7]$.

► Consider function $f : [-1, \infty) \to [0, \infty)$, $f(x) = \begin{cases} x+1, & x \in [-1, 4) \\ \sqrt{x}+3, & x \in [4, \infty). \end{cases}$. Since f is defined from $[-1, \infty)$ to $[0, \infty)$

- the domain is $[-1, \infty)$,
- the co-domain is $[0, \infty)$.

To find the range of f, we consider two cases.

(i) For $x \in [-1, 4)$, $f(x) = x + 1$.

$$\begin{aligned} -1 \leq & \quad x & < 4 \\ -1 + 1 \leq & \quad x + 1 & < 4 + 1 \\ 0 \leq & \quad \underbrace{x + 1}_{f(x)} & < 5 \end{aligned}$$

That is, $f(x) = x + 1 \in [0, 5)$

(i) For $x \in [4, \infty)$, $f(x) = \sqrt{x} + 3$.

$$\begin{aligned} 4 \leq & \quad x \\ 2 \leq & \quad \sqrt{x} \\ 2 + 3 \leq & \quad \underbrace{\sqrt{x} + 3}_{f(x)} \end{aligned}$$

That is, $f(x) = \sqrt{x} + 3 \in [5, \infty)$.

From (i) and (ii), the range of f is $[0, 5) \cup [5, \infty) = [0, \infty)$

So,

- ▶ For g, the domain is $D_g = [0, 3)$, the co-domain is $\mathbb{R}$, and the range is $R_g = (\frac{1}{4}, 1]$.
- ▶ For h, the domain is $D_h = [0, 3)$, the co-domain is $\mathbb{R}$, and the range is $R_h = (-2, 7]$.
- ▶ For f, the domain is $D_f = [-1, \infty)$, the co-domain is $[0, \infty)$, and the range is $R_f = [0, \infty)$.

(b) Find the composite functions $f \circ g$, $g \circ f$, and $f \circ h$ together with their domains and ranges.

▶ Find $f \circ g$.

We first have to check that $D_f \cap R_g = [-1, \infty) \cap (\frac{1}{4}, 1] = (\frac{1}{4}, 1] \neq \emptyset$. Note that we know that the range of g is $(\frac{1}{4}, 1]$, so we will use the first condition of f for $x \in [-1, 4)$, so that $f(x) = x + 1$. So we can define the composite function $f \circ g$ as follows

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x+1}\right) = \frac{1}{x+1} + 1.$$

Since $R_g = (\frac{1}{4}, 1]$ and $D_f = [-1, \infty)$, so $R_g \subset D_f$ and we have the domain of $f \circ g$ is the same as $D_g = [0, 3)$. Since, for $x \in [0, 3)$, from (a)

$$\frac{1}{4} < \frac{1}{x+1} \leq 1 \quad \Longrightarrow \quad \frac{1}{4} + 1 < \frac{1}{x+1} + 1 \leq 1 + 1$$

The range of $f \circ g$ is $(\frac{5}{4}, 2)$.

Hence, we can construct the composite function $f \circ g$ defined as $\boxed{(f \circ g)(x) = \frac{1}{x+1} + 1}$ with the domain $[0,3)$ and range $(\frac{5}{4},2)$. ■

► Find $g \circ f$.

We first have to check that $D_g \cap R_f = [0,3) \cap [0,\infty) = [0,3) \neq \emptyset$. So we can construct $g \circ f$, for $x \in D_f = [-1,\infty)$,

$$(g \circ f)(x) = g(f(x))$$

and we also need to have

$$f(x) \in D_g = [0,3), \quad \Leftrightarrow \quad 0 \leq f(x) < 3$$

which will happen only when $x \in [-1,4)$ by using (i) in part (a). That is, we will use $f(x) = x+1$

$$(g \circ f)(x) = g(f(x)) = g(x+1) = \frac{1}{x+1+1} = \frac{1}{x+2},$$

and in this case with

$$0 \leq f(x) < 3 \quad \Leftrightarrow \quad 0 \leq x+1 < 3 \quad \Leftrightarrow \quad -1 \leq x < 2 \quad \Leftrightarrow \quad x \in [-1,2).$$

Hence, the domain of $g \circ f$ is $[-1,2) \cap D_f = [-1,2)$. To find the range, consider $0 \leq f(x) < 3 \quad \Leftrightarrow \quad 0 \leq x+1 < 3 \quad \Leftrightarrow \quad 1 \leq x+2 < 4$ or

$$\frac{1}{4} < \underbrace{\frac{1}{x+2}}_{(g \circ f)(x)} \leq 1.$$

That is, the range is $(\frac{1}{4},1)$. Hence, we can construct the composite function $g \circ f$ defined as $\boxed{(g \circ f)(x) = \frac{1}{x+2}}$ with the domain $[-1,2)$ and range $(\frac{1}{4},1)$. ■

► Find $f \circ h$.

First note that, from (a),

- for the function h, the domain is $D_h = [0,3)$ and the range is $R_h = (-2,7]$.
- for the function f, the domain is $D_f = [-1,\infty)$ and the range is $R_f = [0,\infty)$.

We next have to check that $D_f \cap R_h = [-1,\infty) \cap (-2,7] = [-1,7] \neq \emptyset$. So we can construct $f \circ h$, for $x \in D_h = [0,3)$,

$$(f \circ h)(x) = f(h(x)) = f\left(\frac{12}{x+1} - 5\right).$$

There are two possible formulae for f, which depend on the value of $h(x) = \frac{12}{x+1} - 5$, i.e. when $h(x) \in [-1,4)$ and when $h(x) \in [4,\infty)$. Since the range of $h(x)$ is $R_h = (-2,7]$, these two cases become the followings.

(1) $h(x) \in [-1,4) \cap (-2,7] = [-1,4)$: For $h(x) \in [-1,4)$,

$$(f \circ h)(x) = f(h(x)) = f\left(\frac{12}{x+1} - 5\right) = \frac{12}{x+1} - 5 + 1 = \frac{12}{x+1} - 4.$$

The value of x for this condition of $(f \circ h)(x)$ must satisfy:

$$\begin{aligned}
-1 \leq & \quad h(x) & < 4 \\
-1 \leq & \quad \tfrac{12}{x+1} - 5 & < 4 \\
\frac{-1+5}{12} \leq & \quad \tfrac{1}{x+1} & < \frac{4+5}{12} \qquad \Rightarrow \frac{1}{3} \leq \frac{1}{x+1} < \frac{3}{4} \\
\frac{4}{3} < & \quad x+1 & \leq 3 \\
\frac{1}{3} < & \quad x & \leq 2
\end{aligned}$$

That is, $\boxed{(f \circ h)(x) = \frac{12}{x+1} - 4 \text{ for } x \in (1/3, 2]}$.
Since $-1 \leq h(x) < 4$ or $-1 \leq \frac{12}{x+1} - 5 < 4$, we have

$$-1 + 1 \leq \frac{12}{x+1} - 5 + 1 < 4 + 1,$$

or $(f \circ h)(x) = \frac{12}{x+1} - 4 \in [0, 5)$ in this case.

(2) $\underline{h(x) \in [4, \infty) \cap (-2, 7] = [4, 7]}$: For $h(x) \in [4, 7]$,

$$(f \circ h)(x) = f(h(x)) = f\left(\frac{12}{x+1} - 5\right) = \sqrt{\frac{12}{x+1} - 5} + 3.$$

The value of x for this condition of $(f \circ h)(x)$ must satisfy:

$$\begin{aligned}
4 \leq & \quad h(x) & \leq 7 \\
4 \leq & \quad \tfrac{12}{x+1} - 5 & \leq 7 \\
\frac{4+5}{12} \leq & \quad \tfrac{1}{x+1} & \leq \frac{7+5}{12} \qquad \Rightarrow \frac{3}{4} \leq \frac{1}{x+1} \leq 1 \\
1 \leq & \quad x+1 & \leq \frac{4}{3} \\
0 \leq & \quad x & \leq \frac{1}{3}
\end{aligned}$$

That is, $\boxed{(f \circ h)(x) = \sqrt{\frac{12}{x+1} - 5} + 3 \text{ for } x \in [0, 1/3]}$.
Since $4 \leq h(x) \leq 7$ or $4 \leq \frac{12}{x+1} - 5 \leq 7$,

$$\sqrt{4} + 3 \leq \sqrt{\frac{12}{x+1} - 5} \leq \sqrt{7} + 3, \quad \Rightarrow \quad 5 \leq (f \circ h)(x) \leq \sqrt{7} + 3,$$

we have $(f \circ h)(x) = \sqrt{\frac{12}{x+1} - 5} + 3 \in [5, \sqrt{7} + 3]$

From (1) and (2), the composite function $f \circ h$ can be written as follow:

$$(f \circ h)(x) = \begin{cases} \sqrt{\frac{12}{x+1} - 5} + 3, & x \in [0, \frac{1}{3}] \\ \frac{12}{x+1} - 4, & x \in (\frac{1}{3}, 2], \end{cases}$$

with domain $[0, \frac{1}{3}] \cup (\frac{1}{3}, 2] = [0, 2]$
and range $[0, 5) \cup [5, \sqrt{7} + 3] = [0, \sqrt{7} + 3]$. ■

3. Define a function $f : [-1, \infty) \to [0, \infty)$ be a function

$$f(x) = \begin{cases} x + 1, & x \in [-1, 4) \\ \sqrt{x} + 3, & x \in [4, \infty). \end{cases}$$

Show that f is injective and find its inverse function f^{-1}.

<u>Answer:</u> To show that f is injective, we will show that f is one-to-one and onto.

(I) Show that $f : [-1, \infty) \to [0, \infty)$ is one-to-one. To show that f is one-to-one, consider the definition (and the version of its contrapositive). For any $x_1, x_2 \in X$,

$$(*) \quad \text{if} \quad f(x_1) = f(x_2), \quad \text{then} \quad x_1 = x_2$$

and the equivalent definition from its contrapositive

$$(**) \quad \text{if} \quad x_1 \neq x_2 \quad \text{then} \quad f(x_1) = f(x_2).$$

Let $x_1, x_2 \in [-1, \infty)$. Then there are 3 cases.

- <u>Case 1:</u> For $x_1, x_2 \in [-1, 4)$, then $f(x) = x + 1$. From $(*)$, suppose $f(x_1) = f(x_2)$.

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 + 1 = x_2 \quad \Rightarrow \quad x_1 = x_2.$$

- <u>Case 2:</u> For $x_1, x_2 \in [4, \infty)$, then $f(x) = \sqrt{x} + 3$. From $(*)$, suppose $f(x_1) = f(x_2)$.

$$f(x_1) = f(x_2) \quad \Rightarrow \quad \sqrt{x_1} + 3 = \sqrt{x_2} + 3 \quad \Rightarrow \quad \sqrt{x_1} = \sqrt{x_2} \quad \Rightarrow \quad x_1 = x_2.$$

- <u>Case 3:</u> For $x_1 \in [-1, 4)$ and $x_2 \in [4, \infty)$, then $f(x_1) = x_1 + 1$ and $f(x_2) = \sqrt{x_1} + 3$. We will use the definition $(**)$ using contrapositive of $(*)$. Since x_1 and x_2 are from two disjoint intervals (i.e. $[0, 4) \cap [4, \infty) = \emptyset$), $x_1 \neq x_2$ and

$$x_1 \in [-1, 4) \quad \Rightarrow \quad -1 \leq x_1 < 4 \quad \Rightarrow \quad 0 \leq x_1 + 1 < 5 \quad \Rightarrow \quad f(x_1) \in [0, 5),$$

$$x_2 \in [4, \infty) \quad \Rightarrow \quad x_2 > 4 \quad \Rightarrow \quad \sqrt{x_2}+3 < \sqrt{4}+3 \quad \Rightarrow \quad f(x_2) < 5 \quad \Rightarrow \quad f(x_2) \in [5, \infty).$$

Notice that $f(x_1)$ and $f(x_2)$ are from disjoint intervals (i.e. $[0, 5) \cap [5, \infty) = \emptyset$). That is, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Note that a similar argument can be done when switching the roles of x_1 and x_2: $x_2 \in [-1, 4)$ and $x_1 \in [4, \infty)$.

From cases 1-3, f is one-to-one.

(II) Show that $f : [-1, \infty) \to [0, \infty)$ is onto.

It is useful to find the range of f first.
(i) When $x \in [-1, 4)$, $f(x) = x + 1$

$$x \in [-1, 4) \quad \Rightarrow \quad -1 \leq x < 4 \quad \Rightarrow \quad 0 \leq x + 1 < 5 \quad \Rightarrow \quad f(x) \in [0, 5).$$

(ii) When $x \in [4, \infty)$, $f(x) = \sqrt{x} + 3$ and

$$x \in [4, \infty) \quad \Rightarrow \quad x > 4 \quad \Rightarrow \quad \sqrt{x} + 3 < \sqrt{4} + 3 \quad \Rightarrow \quad f(x) < 5 \quad \Rightarrow \quad f(x) \in [5, \infty).$$

So the range is $[0, 5) \cup [5, \infty) = [0, \infty)$.
Notice that the range is the same as the co-domain. To show that f is onto. Let $y \in [0, \infty)$. From (i) and (ii), we will consider 2 cases.
- Case 1: For $y \in [0, 5)$, we use (i) and see that if $y = f(x)$, we must have $f(x) = x + 1$. Set $y = x + 1$ Then $\boxed{x = y - 1}$ and
$0 \leq y < 5 \quad \Rightarrow \quad -1 \leq y - 1 < 4$.
So, for any given $y \in [0, 5)$, x that makes $y = f(x)$ is in $[-1, 4)$, which is inside the domain of f.
- Case 2: $y \in [5, \infty)$, we use (ii) and see that if $y = f(x)$, we must have $f(x) = \sqrt{x} + 3$. Set $y = \sqrt{x} + 3$. Then $\boxed{x = (y - 3)^2}$ and
$y \geq 5 \quad \Rightarrow \quad y - 3 \geq 2 \quad \Rightarrow \quad (y - 3)^2 \geq 4$.
So, for any given $y \in [5, \infty)$, x that makes $y = f(x)$ is in $[4, \infty)$, which is inside the domain of f.

From cases 1-2, we covered all possible values of y in the co-domain $[0, \infty)$, so f is onto.

From (I) and (II), f is one-to-one and onto. That is, f is bijective.

Therefore, the inverse function f^{-1} exists. The cases 1 and 2 in (II) give us the formula for $f^{-1} : [0, \infty) \to [-1, \infty)$ as shown below:

$$f^{-1}(y) = \begin{cases} y - 1, & y \in [0, 5) \\ (y - 3)^2, & y \in [5, \infty). \end{cases}$$

Equivalently, we can also write this in term of variable x as $f^{-1}(x) = \begin{cases} x - 1, & x \in [0, 5) \\ (x - 3)^2, & x \in [5, \infty). \end{cases}$

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