

Chapter 9

Optimization with Equality Constraint with many choice variables

 Lagrange multiplier
 Conditions for optimization
 Maximize output level subject to cost constraint
 Minimize cost subject to output constraint
 ~~maximize~~ Minimize utility subject to budget constraint
 Minimize total expenditure under the level of utility



Transition of idea

Ch. 8
Unconstrained optimization

$$\max/\min_{x_1, x_2} z = f(x_1, x_2)$$

Ch. 9
Constrained optimization

$$\max/\min_{x_1, x_2} z = f(x_1, x_2) \\ \text{s.t. } h(x_1, x_2) = C \Rightarrow \text{constraint}$$

Lagrangian function

$$\max/\min_{x_1, x_2, \lambda} L = f(x_1, x_2) + \lambda [C - h(x_1, x_2)]$$

as if λ becomes another choice variable

FOC partial derivative w.r.t. every choice variables

SOC matrix that collects every possible 2nd order partial derivative of obj fn / Lagrangian fn
Hessian Matrix Bordered Hessian matrix

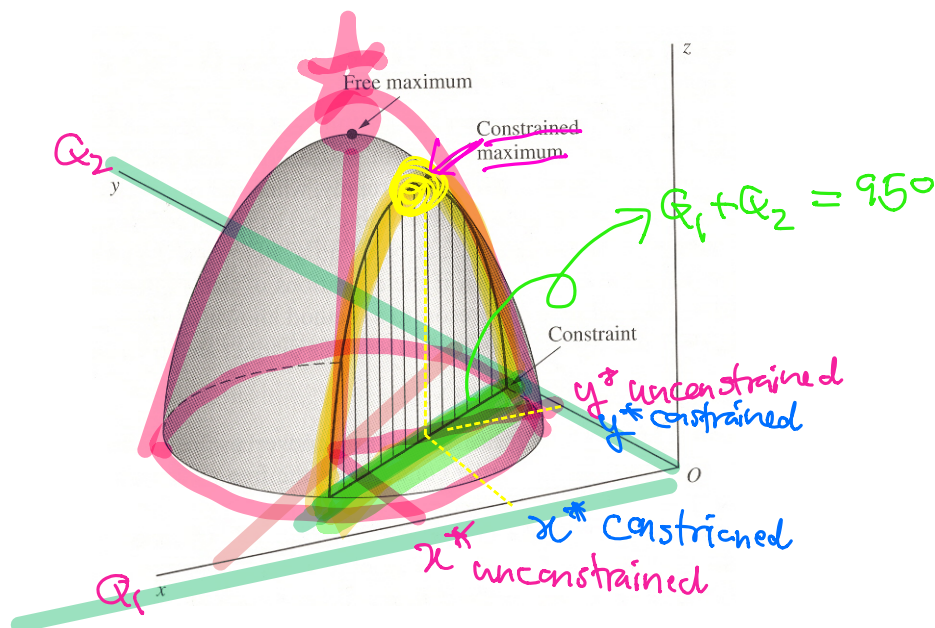
What if choice variables cannot be chosen freely? For example, what if a firm cannot freely choose quantities because of production quota? Firm will need to find constrained optimum value.

Suppose that a firm has two factories and is subjected to production quota of 950. The profit maximization problem becomes:

$$\begin{aligned} & \max_{Q_1, Q_2} \pi(Q_1, Q_2) \\ & \text{Subject to } Q_1 + Q_2 = 950 \end{aligned}$$

,where Q_1, Q_2 are quantity produced at factory 1 and 2, respectively.

Free Extremum vs. Constrained Extremum



Economics is about finding the optimal allocation of scarce resources. Hence, the main problems in economics can be studied as constrained optimization problem. The prototype of such problem is:

$$\begin{array}{l} \text{maximize } f(x_1, x_2, \dots, x_n) \\ \text{where } (x_1, x_2, \dots, x_n) \text{ must satisfy} \end{array}$$

$$g_1(x_1, x_2, \dots, x_n) \leq b_1$$

$$g_2(x_1, x_2, \dots, x_n) \leq b_2$$

.....

$$g_n(x_1, x_2, \dots, x_n) \leq b_n$$

$$h_1(x_1, x_2, \dots, x_n) = c_1$$

$$h_2(x_1, x_2, \dots, x_n) = c_2$$

.....

$$h_n(x_1, x_2, \dots, x_n) = c_n$$

inequality constraint

equality constraint

What will be focusing on?



Optimization with Equality Constraints : Two variables and One Equality Constraint

Let Mr. Musk have utility function $U = x_1x_2 + 2x_1$. If $P_1 = 4$ and $P_2 = 2$

Baht per piece, how many piece of good 1 and good 2 will Mr. Musk choose to consume to maximize his utility?

Suppose that Mr. Musk has no budget constraint

$$\text{Max}_{x_1, x_2} U = x_1x_2 + 2x_1$$

$$\text{FOC: } U_{x_1} = x_2 + 2 = 0 \rightarrow x_2^* = -2$$

$$U_{x_2} = x_1 = 0 \rightarrow x_1^* = 0$$

$$\text{Sec: } H = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|H_1| = |0| = 0 \neq 0$$

$$|H_2| = -1 \neq 0$$

H is indefinite

x_1^*, x_2^* don't give $U_{\max}$

$$U_{x_1} = MU_{x_1} = x_2 + 2$$

$\Rightarrow$

$$\frac{\partial MU_{x_1}}{\partial x_1} = 0$$

$$U_{x_2} = MU_{x_2} = x_1$$

$\Rightarrow$

$$\frac{\partial MU_{x_2}}{\partial x_2} = 0$$

Law of diminishing MU doesn't apply here.

Now suppose that Mr. Musk has the budget equal to 60 baht, how many piece of good 1 and good 2 will Mr. Musk choose to consume to maximize his utility?

Mr. Musk's budget const is $4x_1 + 2x_2 = 60$

$$\begin{aligned} 2x_2 &= 60 - 4x_1 \\ \hookrightarrow x_2 &= 30 - 2x_1 \end{aligned}$$

$$\max_{x_1} U = x_1(30 - 2x_1) + 2x_1$$

"Lagrange Multiplier"

Let f and h be functions with two independent variables x_1, x_2

$$\begin{aligned} &\max_{x_1, x_2} f(x_1, x_2) \\ &\text{Subject to } h(x_1, x_2) = c \end{aligned} \quad \rightarrow \quad c - h(x_1, x_2) = 0$$

"Lagrangian Function"

$$\max_{x_1, x_2, \mu} L(x_1, x_2, \mu) = f(x_1, x_2) + \mu [c - h(x_1, x_2)]$$

↑ lagrange multiplier

We will call μ Lagrange multiplier.

Treat μ as if it is another choice variable.

When the constraint is satisfied, $c - h(x_1, x_2) = 0$,

maximizing/minimizing lagrangian f^b is equivalent to
" _____ " the objective f^a

FONC: choice vars are x_1, x_2, μ

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= f_1 - \mu h_1(x_1, x_2) = 0 \\ \frac{\partial L}{\partial x_2} &= f_2 - \mu h_2(x_1, x_2) = 0 \\ \frac{\partial L}{\partial \mu} &= c - h(x_1, x_2) = 0 \end{aligned} \quad \left. \begin{array}{l} \text{w.r.t. every choice var} \\ \text{w.r.t. } \mu \end{array} \right\} \begin{array}{l} \text{solve for} \\ x_1^*, x_2^*, \\ \mu^* \\ \text{that satisfy} \\ \text{all FONCs.} \end{array}$$

Mr. Musk's problem is

$$\begin{aligned} &\max_{x_1, x_2} U = x_1 x_2 + 2x_1 \\ &\text{st. } h(x_1, x_2) = c \\ &\quad 4x_1 + 2x_2 = 60 \\ &\quad \hookrightarrow 60 - 4x_1 - 2x_2 = 0 \end{aligned}$$

Lagrangian function for utility maximization problem when Mr. Musk has 60 baht is:

$$\max_{x_1, x_2, \mu} L = x_1 x_2 + 2x_1 + \mu [60 - 4x_1 - 2x_2]$$

these are there because of we consider info. from the constraint.

FONC:

$$L_\mu = \frac{\partial L}{\partial \mu} = 60 - 4x_1 - 2x_2 = 0 \Rightarrow 4x_1 + 2x_2 = 60 \quad (1)$$

$$L_1 = \frac{\partial L}{\partial x_1} = x_2 + 2 - 4\mu = 0 \Rightarrow x_2 + 2 = 4\mu \quad (2)$$

$$L_2 = \frac{\partial L}{\partial x_2} = x_1 - 2\mu = 0 \Rightarrow x_1 = 2\mu \quad (3)$$

$$(2) / (3) \quad \frac{x_2 + 2}{x_1} = \frac{4\mu}{2\mu} = 2 \Rightarrow x_1 = \frac{x_2 + 2}{2} \quad (4)$$

$$(4) \text{ in } (1) \quad 4 \left(\frac{x_2 + 2}{2} \right) + 2x_2 = 60$$

$$x_2^* = 14$$

$$x_1^* = 8$$

$$\mu^* = \frac{x_1^*}{2} = 4$$

Homework: Write Lagrangian function, first order conditions, and critical values of the following problems:

(a.)

$$\text{Max } U(x_1, x_2) = x_1 x_2$$

$$\text{St. } h(x_1, x_2) = x_1 + 4x_2 = 16$$

(b.)

$$\text{Max } z = xy$$

$$\text{St. } x + y = 6$$



Optimization with Equality Constraints : n variables and One Equality Constraint

$$\max_{x_1, \dots, x_n} f(x_1, x_2, \dots, x_n)$$

$$\text{st. } h(x_1, x_2, \dots, x_n) = c$$

Lagrangian function is

$$\max_{x_1, \dots, x_n, \mu} L = f(x_1, \dots, x_n) + \mu [c - h(x_1, \dots, x_n)]$$

= 0 if the const. is satisfied.

$$\begin{aligned} \text{FOC : } L_\mu &= \frac{\partial L}{\partial \mu} = c - h(x_1, \dots, x_n) = 0 \\ L_1 &= \frac{\partial L}{\partial x_1} = f_1 - \mu h_1 = 0 \\ L_2 &= \frac{\partial L}{\partial x_2} = f_2 - \mu h_2 = 0 \\ &\vdots \\ L_n &= \frac{\partial L}{\partial x_n} \end{aligned} \quad \left. \begin{array}{l} \text{Use } n+1 \text{ FOCs} \\ \text{to solve for} \\ \text{critical values of} \\ x_1^*, x_2^*, \dots, x_n^*, \mu^* \end{array} \right\}$$

Note 1 : At critical values, the constraint is satisfied,

$$L^* = f(x_1^*, x_2^*, \dots, x_n^*) + \mu^* [c - h(x_1^*, x_2^*, \dots, x_n^*)]$$

$$L^* = f(x_1^*, x_2^*, \dots, x_n^*) \quad \mu^* = 0$$

The value of Lagrangian f^* is equal to the value of obj f^* at the critical point.

$$\text{Note 2 : } \frac{\partial L^*}{\partial c} = \mu^* = \frac{\partial f(x_1^*, \dots, x_n^*)}{\partial c}$$

shows how optimized value of objective f^* changes when constraint c changes 1 unit

ex. when budget increases by 1 bcht, how much will the maximized utility change? $\Rightarrow$ it would change by μ^* utils.

Second-Order Sufficient Condition

Consider objective function:

$$\begin{aligned} \text{Max}_{x_1, \dots, x_n} & f(x_1, x_2, \dots, x_n) \\ \text{St.} & \begin{aligned} h^1(x_1, x_2, \dots, x_n) &= c_1 \\ h^2(x_1, x_2, \dots, x_n) &= c_2 \\ &\dots \\ h^k(x_1, x_2, \dots, x_n) &= c_k \end{aligned} \end{aligned}$$

} k equality constraints.
k lagrange multipliers.

The Lagrangian

$$\text{max}_{x_1, \dots, x_n, \mu_1, \dots, \mu_k} L(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + \mu_1 [c_1 - h^1(x_1, x_2, \dots, x_n)] + \mu_2 [c_2 - h^2(x_1, x_2, \dots, x_n)] + \dots + \mu_k [c_k - h^k(x_1, x_2, \dots, x_n)]$$

} n+k terms

} n choice vars k lagrange multipliers

First Order Necessary Condition: FONC

$$\left. \begin{aligned} L_{x_1} = \frac{\partial L}{\partial x_1} = 0 & \quad f_1 - \mu_1 h_1^1 - \mu_2 h_1^2 - \dots - \mu_k h_1^k = 0 \\ \vdots \\ L_{x_n} = \frac{\partial L}{\partial x_n} = 0 & \quad f_n - \mu_1 h_n^1 - \mu_2 h_n^2 - \dots - \mu_k h_n^k = 0 \\ \vdots \\ L_{\mu_1} = \frac{\partial L}{\partial \mu_1} = 0 & \quad c_1 - h^1(x_1, \dots, x_n) = 0 \\ \vdots \\ L_{\mu_k} = \frac{\partial L}{\partial \mu_k} = 0 & \quad c_k - h^k(x_1, \dots, x_n) = 0 \end{aligned} \right\} \begin{array}{l} n+k \text{ eqs.} \\ k \text{ eqs.} \end{array}$$

Second Order Sufficient Condition: SOSOC

"Bordered Hessian Matrix"..... $\bar{H}$: a matrix collecting every possible 2nd order partial derivatives of the lagrangian fⁿ

1st order partial derivatives: $L_{\mu_1}, L_{\mu_2}, \dots, L_{\mu_k}, L_{x_1}, L_{x_2}, \dots, L_{x_n}$: n+k terms.
Take partial diff of 1st order w.r.t. $\mu_1, \dots, \mu_k, x_1, \dots, x_n$
We can divide Bordered Hessian Matrix into 4 parts:

Note: We evaluate Bordered Hessian at the critical values.

partial diff w.r.t. μ_1
 μ_2
 μ_k
 x_1
 x_2
 x_n

row 1
 row 2
 row k
 row k+1
 row n

Col 1
 Col 2
 ...
 Col k
 ...
 Col k+1
 Col k+2
 ...
 Col k+n

$L_{\mu_1 \mu_1}$ $L_{\mu_1 \mu_2}$... $L_{\mu_1 \mu_k}$ $L_{\mu_1 x_1}$ $L_{\mu_1 x_2}$... $L_{\mu_1 x_n}$
 $L_{\mu_2 \mu_1}$ $L_{\mu_2 \mu_2}$... $L_{\mu_2 \mu_k}$ $L_{\mu_2 x_1}$ $L_{\mu_2 x_2}$... $L_{\mu_2 x_n}$
 $\vdots$ $\vdots$ $\ddots$ $\vdots$ $\vdots$ $\vdots$
 $L_{\mu_k \mu_1}$ $L_{\mu_k \mu_2}$... $L_{\mu_k \mu_k}$ $L_{\mu_k x_1}$ $L_{\mu_k x_2}$... $L_{\mu_k x_n}$
 $L_{x_1 \mu_1}$ $L_{x_1 \mu_2}$... $L_{x_1 \mu_k}$ L_{11} L_{12} ... L_{1n}
 $L_{x_2 \mu_1}$ $L_{x_2 \mu_2}$... $L_{x_2 \mu_k}$ L_{21} L_{22} ... L_{2n}
 $\vdots$ $\vdots$ $\ddots$ $\vdots$ $\vdots$ $\ddots$ $\vdots$
 $L_{x_n \mu_1}$ $L_{x_n \mu_2}$... $L_{x_n \mu_k}$ L_{n1} L_{n2} ... L_{nn}

$\bar{H}$
 $(n+k) \times (n+k)$

$L_{\mu_i \mu_j} = 0 \quad i=1, \dots, k, j=1, \dots, k$

taking partial diff of the first const w.r.t. $x_1, \dots, x_n$ and with negative sign

$\bar{H} = \begin{bmatrix} 0 & 0 & \dots & 0 & -h_1^1 & -h_2^1 & \dots & -h_n^1 \\ 0 & 0 & \dots & 0 & -h_1^2 & -h_2^2 & \dots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -h_1^k & -h_2^k & \dots & -h_n^k \\ -h_1^1 & -h_1^2 & \dots & -h_1^k & L_{11} & L_{12} & \dots & L_{1n} \\ -h_2^1 & -h_2^2 & \dots & -h_2^k & L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \dots & -h_n^k & L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix}$
 $(n+k) \times (n+k)$

We can have submatrix H_1 with border, $\bar{H}_1$, and so on. Try to get the submatrix from $\bar{H}$.

$\bar{H}_1$ is H_1 including all rows above & all column to the left

$$\bar{H}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\ 0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\ -h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ -h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}_{(n+k) \times (n+k)}$$

$\bar{H}_2$

$$\bar{H}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\ 0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\ -h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ -h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}_{(n+k) \times (n+k)}$$

Question:

How many submatrices do we need to consider?

Answer:

$n - k$ matrices, i.e. the last $n-k$ leading principle minors of matrix $\bar{H}$.

n is the number of choice variables and k is the number of constraints.

We need to focus since $\bar{H}_{k+1}$.

We need to check:

$$\bar{H}_{k+1}, \bar{H}_{k+2}, \dots, \bar{H}_n = \bar{H}$$

$\bar{H}_1, \bar{H}_2, \bar{H}_3, \dots, \bar{H}_k, \bar{H}_{k+1}, \dots, \bar{H}_n$

drop off from our consideration

consider the last $n-k$ submatrices

Example 1: the case of one constraint with two choice variables:

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1 x_1} & L_{x_1 x_2} \\ -h_{x_2} & L_{x_2 x_1} & L_{x_2 x_2} \end{bmatrix}$$

3×3
 $(2+1) \times (2+1)$

Number of choice variables (n) 2

Number of constraint (k) 1

So we need to check the last $n-k = \dots\dots\dots 2-1 = 1$ Matrix, which is:

$$\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1 x_1} & L_{x_1 x_2} \\ -h_{x_2} & L_{x_2 x_1} & L_{x_2 x_2} \end{bmatrix} = \bar{H}$$

Example 2: 3 choice variables and 1 constraint

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1 x_1} & L_{x_1 x_2} & L_{x_1 x_3} \\ -h_{x_2} & L_{x_2 x_1} & L_{x_2 x_2} & L_{x_2 x_3} \\ -h_{x_3} & L_{x_3 x_1} & L_{x_3 x_2} & L_{x_3 x_3} \end{bmatrix} \quad (3+1) \times (3+1)$$

Diagram illustrating the selection of the matrix $\bar{H}_2$ for the Hessian check. The matrix $\bar{H}$ is shown with dimensions $(3+1) \times (3+1)$. The first column and first row are crossed out (labeled $\bar{H}_1$ and "drop"). The remaining 3×3 submatrix is labeled $\bar{H}_2$ and "keep".

Number of choice variables (n) 3

Number of constraint (k) 1

keep the last $n-k$, drop the first k

So we need to check the last $n-k = \dots\dots\dots 3-1 = 2$ Matrix, beginning at $\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2$

$$\bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1 x_1} & L_{x_1 x_2} \\ -h_{x_2} & L_{x_2 x_1} & L_{x_2 x_2} \end{bmatrix} \quad \bar{H}_3 = \bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1 x_1} & L_{x_1 x_2} & L_{x_1 x_3} \\ -h_{x_2} & L_{x_2 x_1} & L_{x_2 x_2} & L_{x_2 x_3} \\ -h_{x_3} & L_{x_3 x_1} & L_{x_3 x_2} & L_{x_3 x_3} \end{bmatrix}$$

We need to observe pattern of the determinants of leading principal submatrices as the following.

The objective function will be at the local max if the last $(n-k)$ leading principal minors have alternate signs with $|\bar{H}_{k+1}|$ has the same sign as $(-1)^{k+1}$.

The objective function will be at the local min if the last $(n-k)$ leading principal minors have the same sign as $(-1)^k$ for every bordered Hessian that we need to consider.

∞ SOC of 2 choice variables + 1 equality constraint ∞

$$|\bar{H}_2| = \begin{vmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1 x_1} & L_{x_1 x_2} \\ -h_{x_2} & L_{x_2 x_1} & L_{x_2 x_2} \end{vmatrix} \quad 3 \times 3$$

If $|\bar{H}_2| > 0 \quad \rightarrow$

If $|\bar{H}_2| < 0 \quad \rightarrow$

$n = 2, k = 1$ (consider x_1, x_2, μ when taking partial diff of Lagrangian f^2)

look at the last $n-k = 2-1 = 1$ matrix

which is $\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2$

Obj. f^2 is maximized when $|\bar{H}_{k+1}|$ has the same sign as $(-1)^{k+1} = (-1)^{1+1} = (-1)^2 = 1 > 0$

$\therefore |\bar{H}_{k+1}| = |\bar{H}_2| > 0$

minimized

has the same sign as

$(-1)^k = (-1)^1 = -1 < 0$

$|\bar{H}_{k+1}| < 0$

" $|\bar{H}_2|$

FOC, SOC and n choice variables + 1 equality constraint

$$\max / \min_{x_1, x_2, \dots, x_n} f(x_1, x_2, \dots, x_n)$$

$$\text{St. } h(x_1, x_2, \dots, x_n) = c$$

$$\text{FOC: } L = f(x_1, x_2, \dots, x_n) + \lambda [c - g(x_1, x_2, \dots, x_n)]$$

$$L_1 = L_2 = L_3 = \dots = L_n = L_\lambda = 0$$

SOC:

★ MAX

$$|\bar{H}_2| > 0, |\bar{H}_3| < 0, |\bar{H}_4| > 0, \dots$$

$|\bar{H}_2|$ has the same sign as $(-1)^{k+1} = (-1)^{1+1} > 0$ and alternate

★ MIN

$$|\bar{H}_2|, |\bar{H}_3|, |\bar{H}_4|, \dots, |\bar{H}_n| < 0$$

$|\bar{H}_2|$ has the same sign as $(-1)^1 = (-1)^1 < 0$

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} & \dots & -h_{x_n} \\ -h_{x_1} & L_{11} & L_{12} & L_{13} & \dots & L_{1n} \\ -h_{x_2} & L_{21} & L_{22} & L_{23} & \dots & L_{2n} \\ -h_{x_3} & L_{31} & L_{32} & L_{33} & \dots & L_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_{x_n} & L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} \end{bmatrix}$$

$(n+1) \times (n+1)$

firm's

$$\begin{array}{ll} \text{Max} & Q \\ K, L & \\ \text{st.} & C \end{array} \quad \begin{array}{ll} \text{Min} & TC \\ K, L & \\ \text{st.} & Q_0 \end{array}$$

Consumer's

$$\begin{array}{ll} \text{Max} & U \\ x, y & \\ \text{st.} & B \end{array} \quad \begin{array}{ll} \text{Min} & \text{Exp} \\ x, y & \\ \text{st.} & U_0 \end{array}$$



Output maximization problem

Cost minimization problem (CMP)

Maximizing output level subject to cost constraint
Let the production function be $Q = q(L, K)$

Utility Maximization Problem (UMP)

Expenditure minimization Problem (EMP)

The total cost is:

$$TC = wL + rK = C$$

Question: How will firm choose L and K to maximize output level, given that the total cost must be C baht.

$$\begin{array}{ll} \text{Max}_{L, K} & Q = q(L, K) \\ \text{st.} & wL + rK = C \end{array}$$

Step 1: Lagrangian Function

$$\text{max}_{\lambda, L, K} Z = q(L, K) + \lambda [C - wL - rK]$$

Step 2: First-Order Condition (FONC):

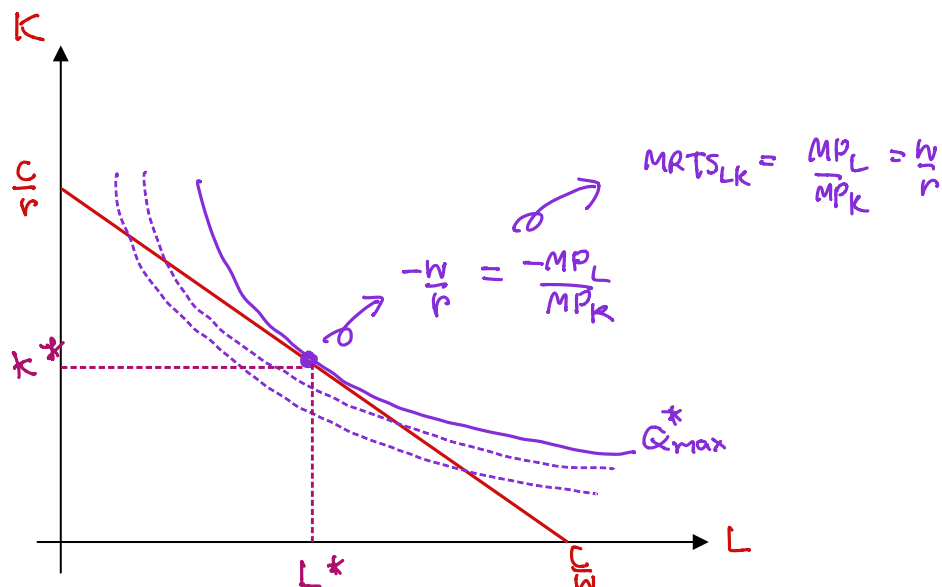
$$\frac{\partial Z}{\partial \lambda} = C - wL - rK = 0 \Rightarrow wL + rK = C \quad (1)$$

$$\frac{\partial Z}{\partial L} = \frac{\partial q(L, K)}{\partial L} - w\lambda = 0 \Rightarrow q_L = w\lambda \quad (2)$$

$$\frac{\partial Z}{\partial K} = \frac{\partial q(L, K)}{\partial K} - r\lambda = 0 \Rightarrow q_K = r\lambda \quad (3)$$

Use (1), (2), (3) to solve for K^*, L^*, λ^*

$$(2)/(3) \text{ MRTS}_{LK} = \frac{q_L}{q_K} = \frac{w}{r}$$

Economic Interpretation

$$\text{isocost : } wL + rK = C \quad \therefore rK = C - wL \\ \Rightarrow K = \frac{C}{r} - \frac{w}{r}L$$

$$\text{isoquant : } Q = q(L, K) = \bar{Q}$$

$$dQ = q_L dL + q_K dK = d\bar{Q} = 0$$

$$\frac{dK}{dL} = -\frac{q_L}{q_K}$$

$$\text{note : } \frac{d^2 K}{dL^2} > 0 \quad \because \frac{d^2 K}{dL^2} = \frac{\partial}{\partial L} \left(-\frac{q_L}{q_K} \right) \frac{dL}{dL} + \frac{\partial}{\partial K} \left(-\frac{q_L}{q_K} \right) \frac{dK}{dL} > 0$$

$$\text{note 2 : } z^* = q(L^*, K^*) + \lambda^* [C - wL^* - rK^*] \\ \frac{\partial z^*}{\partial C} = \lambda^* = \frac{\partial q(L^*, K^*)}{\partial C} \quad \Rightarrow z^* = q(L^*, K^*)$$

$$= \frac{1}{\frac{\partial C}{\partial Q^*}} = \frac{1}{MC^*}$$

$$C \uparrow \quad 1 \text{ baht} \quad \rightarrow \quad Q_{\max}^* \uparrow \quad \lambda^* = \frac{1}{MC^*} \text{ units.}$$

The number of the maximized output that can be increased when budget for cost increases 1 bath.

$$\begin{aligned}\frac{\partial z}{\partial x} &= C - wL - rK \\ \frac{\partial z}{\partial L} &= 2q_L(L,K) - w \\ \frac{\partial z}{\partial K} &= 2q_K(L,K) - r\end{aligned}$$

Step 3: Second-Order Sufficient Condition**3.1 Construct Bordered Hessian**

$$\bar{H} = \begin{bmatrix} z_{\lambda\lambda} = 0 & z_{\lambda L} = -w & z_{\lambda K} = -r \\ z_{L\lambda} = -w & z_{LL} = q_{LL} & z_{LK} = q_{LK} \\ z_{K\lambda} = -r & z_{KL} = q_{KL} & z_{KK} = q_{KK} \end{bmatrix}_{(n+k) \times (n+k)}$$

$$\bar{H} = \begin{bmatrix} 0 & -w & -r \\ -w & q_{LL} & q_{LK} \\ -r & q_{KL} & q_{KK} \end{bmatrix}_{3 \times 3}$$

$\bar{H}_2 = \bar{H}$

3.2 Check the last n-k leading principle Minor

$$n = 2 \quad (K, L) \quad k = 1$$

We need to check the last $n-k$ leading principle Minor, which is $|\bar{H}_{n-k+1}| = |\bar{H}_{1+1}| = |\bar{H}_2|$
 $= 2-1=1$

needs to have the same sign as $(-1)^{k+1} = (-1)^{1+1} = 1 > 0$

$$|\bar{H}_2| = |\bar{H}| > 0$$

$$|\bar{H}| = wrq_{LK} + wrq_{KL} - r^2q_{LL} - w^2q_{KK}$$

$$q_{LK} = q_{KL} > 0$$

if $K \uparrow$, $MP_L \uparrow$
 $L \uparrow$, $MP_K \uparrow$

$$\frac{\partial MP_L}{\partial L}$$

$$< 0$$

$$\frac{\partial MP_K}{\partial K}$$

$$< 0$$

law of diminishing MP_L, MP_K

H.W. Max $Q = L^{0.4} K^{0.5}$ st. $3L + 4K = 108$
 L, K



Minimizing cost given a level of output:

Least Cost Combination of Inputs and Conditional Input Demand

Assume that a firm needs to produce output equal to Q_0 units from production function $Q = F(L, K)$, and the firm would like to choose the level of capital and labor that minimize the cost.

$$\begin{array}{l} \min_{L, K} TC = wL + rK \\ \text{st. } F(L, K) = Q_0 \end{array} \quad \left. \vphantom{\begin{array}{l} \min_{L, K} TC = wL + rK \\ \text{st. } F(L, K) = Q_0 \end{array}} \right\} \text{Cost minimization problem (CMP)}$$

Step 1: Lagrangian Function

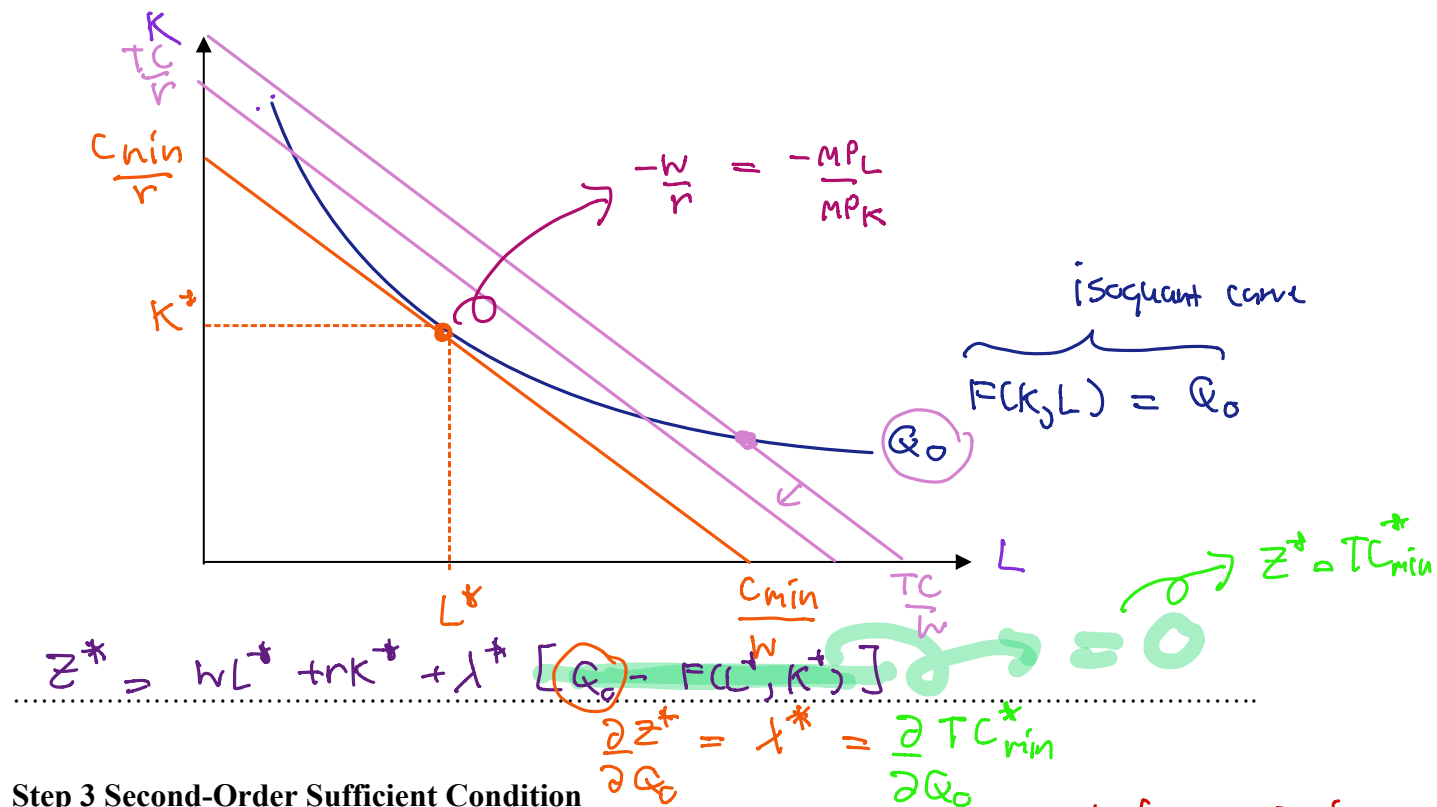
$$\min_{\lambda, L, K} Z = wL + rK + \lambda [Q_0 - F(L, K)]$$

Step 2: First-Order Condition (FONC):

$$\begin{aligned} \frac{\partial Z}{\partial \lambda} &= Q_0 - F(L, K) = 0 \Rightarrow F(L, K) = Q_0 \quad (1.) \\ \frac{\partial Z}{\partial L} &= w - \lambda \frac{\partial F(L, K)}{\partial L} = 0 \Rightarrow \lambda MP_L = w \quad (2.) \\ \frac{\partial Z}{\partial K} &= r - \lambda \frac{\partial F(L, K)}{\partial K} = 0 \Rightarrow \lambda MP_K = r \quad (3.) \end{aligned}$$

(2.) / (3.)

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

Economic Interpretation**Step 3 Second-Order Sufficient Condition****3.1 Construct Bordered Hessian**

$$\bar{H} = \begin{bmatrix} z_{\lambda\lambda} = & z_{\lambda L} = & z_{\lambda K} = \\ z_{L\lambda} = & z_{LL} = & z_{LK} = \\ z_{K\lambda} = & z_{KL} = & z_{KK} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} 0 & -F_L & -F_K \\ -F_L & -\lambda F_{LL} & -\lambda F_{LK} \\ -F_K & -\lambda F_{KL} & -\lambda F_{KK} \end{bmatrix}$$

(2+1) × (2+1)

3.2 Check the last n-k leading principle Minor

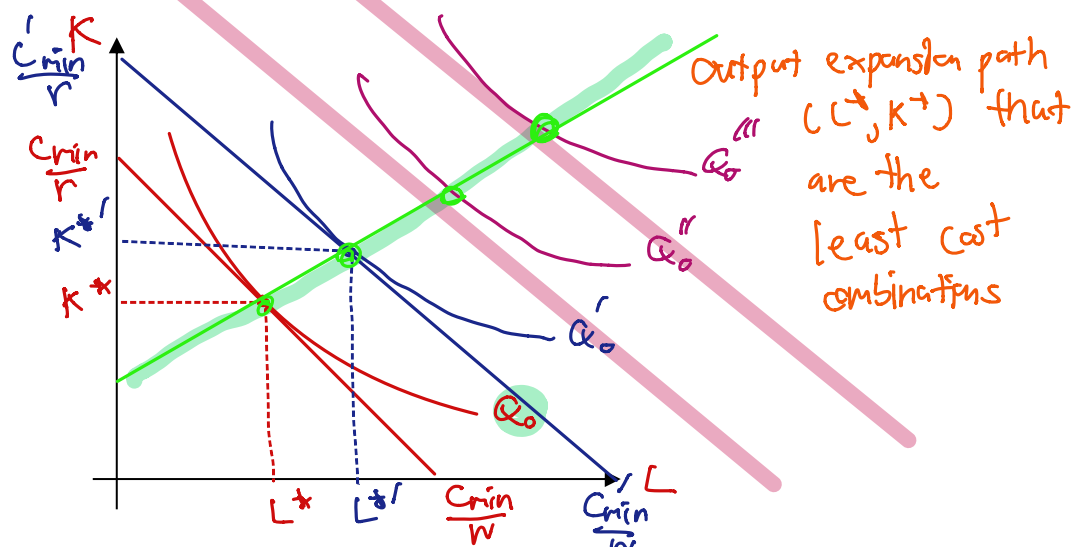
$$n = \underline{2} \quad k = \underline{1}$$

We need to check the last $2-1=1$ leading principle Minor นั่นคือ $|\bar{H}_{k+1}| = |\bar{H}_2| = |\bar{H}|$
with the sign of $(-1)^k = (-1)^1 < 0$

$$|\bar{H}_2| = \underbrace{-\lambda F_{LL} F_{KK}}_{+++} - \underbrace{\lambda F_{LK} F_{KL}}_{\ominus} + \underbrace{\lambda F_{KL}^2}_{\ominus} + \underbrace{\lambda F_{KK}^2}_{\ominus} < 0$$

The Expansion Path

Expansion Path is the least cost combination between L^* and K^* given many levels of given outputs.



If Isoquants are strictly convex, expansion path can be found from the first order conditions.

If a firm has Cobb-Douglas production function $Q = AL^\alpha K^\beta$,

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{w}{r}$$

$$\frac{\alpha A L^{\alpha-1} K^\beta}{\beta A L^\alpha K^{\beta-1}} = \frac{w}{r} \Rightarrow \frac{\alpha K^+}{\beta L^+} = \frac{w}{r}$$

In this case, the optimal ratio of capital to labor is constant and is equal to $\frac{K^+}{L^+} = \frac{\beta w}{\alpha r}$

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Cost minimization problem vs. Profit maximization problem

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Duality

[illegible]

**Utility maximization subject to budget constraint**

$$U = U(x, y) \quad (U_x, U_y > 0)$$

$$\text{St.} \quad xP_x + yP_y = B$$

Step 1: Lagrangian Function

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Step 2: First-Order Condition (FONC):

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Economic Interpretation**Interpretation 1 : Marginal utility per one baht paid**

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Interpretation 2: Indifference Curve

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Meaning of Lagrange multiplier

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Step 3 Second-Order Sufficient Condition**3.1 Construct Bordered Hessian**

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} = & L_{\lambda x} = & L_{\lambda y} = \\ L_{x\lambda} = & L_{xx} = \frac{\partial^2 U}{\partial x^2} = & L_{xy} = \frac{\partial^2 U}{\partial x \partial y} = \\ L_{y\lambda} = & L_{yx} = \frac{\partial^2 U}{\partial y \partial x} = & L_{yy} = \frac{\partial^2 U}{\partial y^2} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n = _____ k = _____

Check the last _____ leading principle Minor :

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Suppose that Mr. Musk's utility function takes the form:

$$U(x, y) = 10x^{\frac{1}{2}}y^{\frac{1}{2}}$$

Mr. Musk has m baht. Price of Good x is P_x , and price of good y is P_y . What will be the level of x and y that maximize the utility?

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Lagrangian function:

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FOC:

[illegible]

SOC:

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Minimize Total Expenditure Under the Level of Utility

Suppose now that a consumer would like to have utility U_0 . Her utility is $U = U(x, y)$.

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Step 1 Lagrangian Function

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Step 3: Second-Order Sufficient Condition**3.1 Construct Bordered Hessian**

$$\bar{H} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n = _____ k = _____

Check the last _____ leading principle Minor:

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Income expansion path (Income consumption curve: ICC)

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Duality: Utility Maximization Problem vs. Expenditure Minimization Problem

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Other optimization problems:

Multiproduct firm with technological constraint

Quantities of two products are x & y .

How many of x & y should the firm choose to produce to minimize the cost of production subjecting to technological constraint?

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Multiplant firm who knows exact total market demand

Let factory 1 produce Q_1 with cost $C_1 = C_1(Q_1)$; factory 2 produce Q_2 with cost $C_2 = C_2(Q_2)$; and let market demand be Q_0 .

To minimize cost, how many should the firm produce at each factory given that it wants to meet the market demand?

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