





CHAPTER 8


Optimization without Constraint: More-Than-One Independent Variable Cases

 Conditions for maximum or minimum

 Third degree price discrimination → multi market

 Competitive Firm Input Choices: Cobb-Douglas Technology → multi factor of production

 Multiproduct-firm → competitive mkt
→ monopoly

 Multiplant-firm

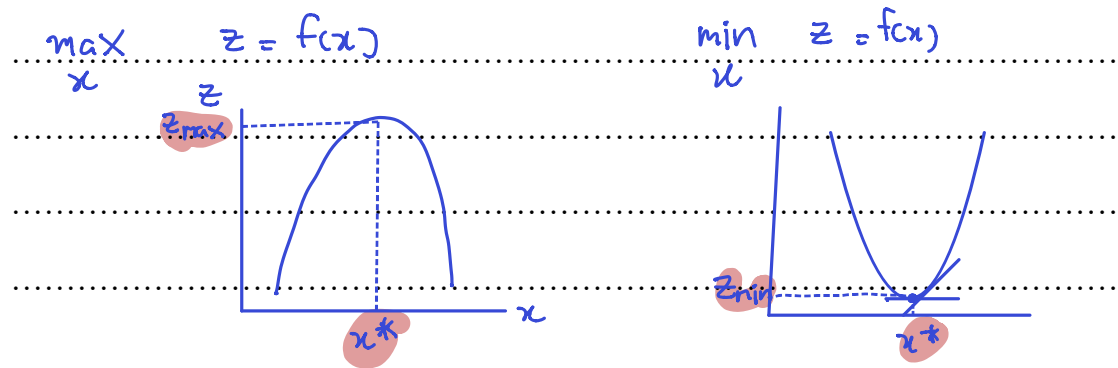


Optimization Condition



One choice variable

The objective function: $z = f(x)$



Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	Derivative $f'(x) = 0$ slope must be zero	Derivative $f'(x) = 0$
	Total differential $dz = f'(x)dx = 0$ $= 0 \quad > 0$	Total differential $dz = f'(x)dx = 0$
	z doesn't change for extremely small increase (or decrease) in x	
Second-order sufficient (SOSC, SOC)	Derivative $f''(x) < 0$ slope is decreasing $(0 \rightarrow -)$	Derivative $f''(x) > 0$ slope is increasing $(0 \rightarrow +)$
$d^2z = d(dz)$ $= d(f'(x)dx)$ $= dx d(f'(x))$ $= dx(f''(x)dx)$ $= f''(x)(dx)^2$	Total differential $d^2z < 0$	Total differential $d^2z > 0$

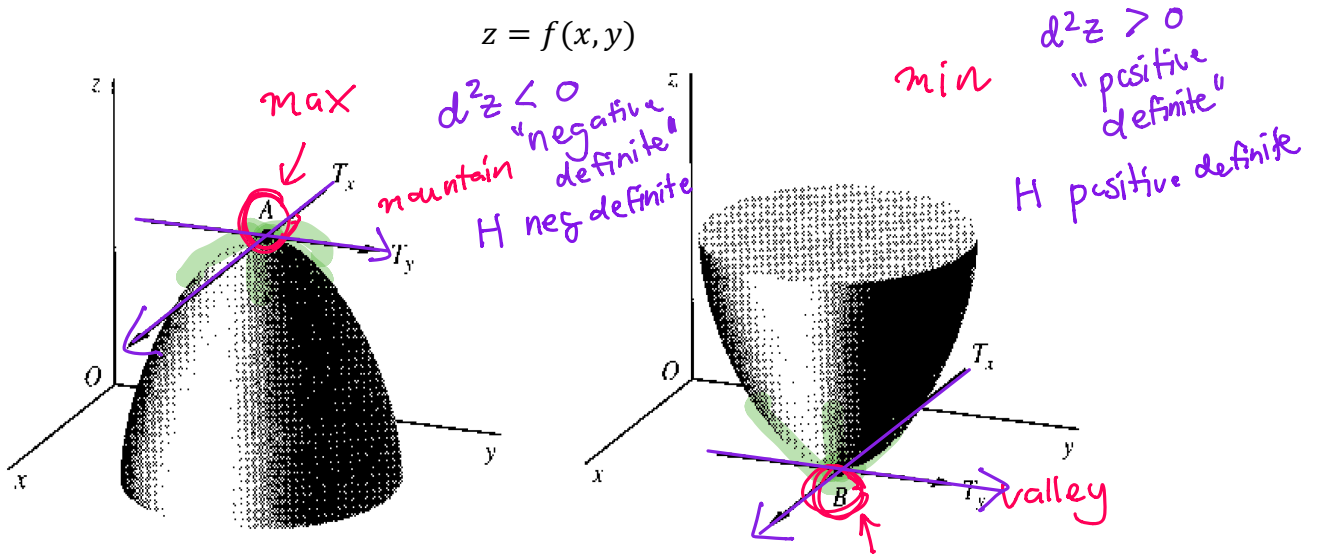
2nd order total differential of z

$$d^2z = f''(x) \frac{dx^2}{> 0}$$



Extreme Value of a Function of Two Variables

$z = f(x, y)$



First-Order Necessary Condition $z = f(x, y)$

$dz = 0$ for every dx and dy

$dz = 0$

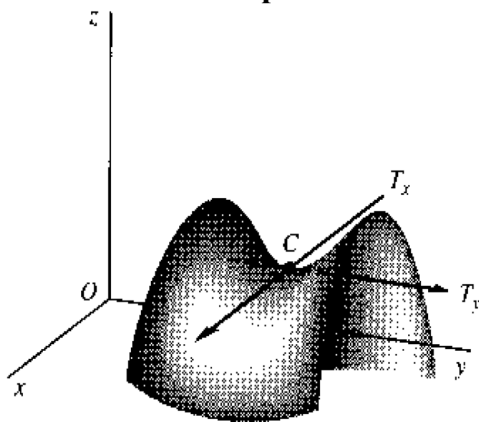
$\star dz = f_x dx + f_y dy = 0; dx, dy > 0$

$f_x = 0, f_y = 0$

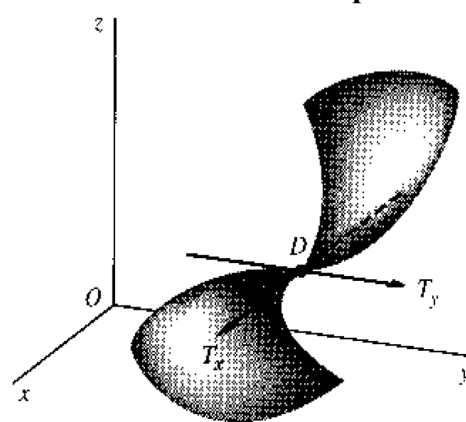
the slope of the tangent line T_x, T_y are zero

The first order necessary condition is necessary, but not sufficient to establish a maximum or a minimum.

Saddle point



Inflection point



Second order sufficient condition

The satisfaction of the first-order condition earmarks certain values of z as the stationary values of the objective function. If at a stationary value of z we find that d^2z is positive definite, i.e., $d^2z > 0$, this will suffice to establish that value of z as a minimum. Analogously, the negative definiteness of d^2z , $d^2z < 0$, is a sufficient condition for the stationary value to be a maximum.

$y = f(x_1, x_2, \dots, x_n)$
 $H_{n \times n}$
 $[f_{ij}]_{n \times n} \Rightarrow H_{n \times n}$

Second-order total differential, Hessian Matrix and Definiteness of Hessian Matrix

📖 About " H_k "

$A_{n \times n}$

Definition : Let A be an $n \times n$ matrix. A $k \times k$ submatrix, deriving from deleting the last $n - k$ columns and the last $n - k$ rows of matrix A , can be called the k^{th} order leading principal submatrix of matrix A . The corresponding determinant of this $k \times k$ submatrix is called the k^{th} order leading principal minor of matrix A .

The k^{th} order leading principal submatrix of matrix A is denoted by A_k .

The k^{th} order leading principal minor of matrix A is denoted by $|A_k|$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 3 \times 3$$

A_1 A_2 A_3

Leading principal submatrices are:

The 1st order leading principal submatrix \Rightarrow delete the last $3 - 1$ rows & columns

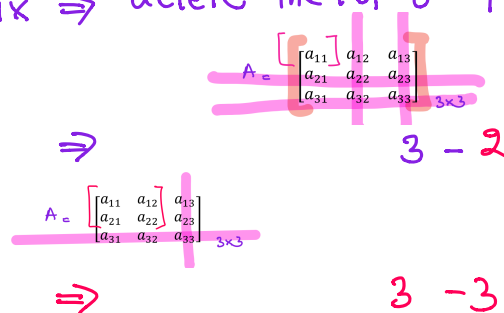
$A_1 = [a_{11}]$

2nd

$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

3rd

$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$



Leading principal minors are:

$$|A_1| = |a_{11}| \quad \text{1st order}$$

$$|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \text{2nd order}$$

$$|A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{3rd order}$$

For Hessian Matrix of function $y = f(x_1, x_2, \dots, x_n)$

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \dots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \dots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \dots & f_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \dots & f_{nn} \end{bmatrix} \quad n \times n$$

$$H_1 = [f_{11}]$$

$$H_2 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \quad \dots \text{delete } n-2 \text{ rows \& column the last}$$

$$H_3 = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \quad \dots \text{delete } n-3 \text{ rows \& column}$$

..... $H_n = H$

Leading principal minors are:

$$|H_1| = |f_{11}| \quad \text{this is not the absolute sign}$$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \quad \dots \quad |H_n| = |H|$$

📖 About "Definiteness of H "

Let H be a symmetric matrix with dimension $n \times n$

$d^2z > 0$ (a) H is **positive definite** if all n leading principal minors are positive
 $|H_1| > 0, |H_2| > 0, |H_3| > 0, \dots$ all positive

$d^2z < 0$ (b) H is **negative definite** if n leading principal minor duly alternate in sign with the first one being negative.

$|H_1| < 0, |H_2| > 0, |H_3| < 0, \dots$ starting with negative, alternate in sign

If (a) or (b) is not met, H is indefinite.

Example: Find definiteness of matrix $B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}_{2 \times 2}$

$$B_1 = [-3], \quad |B_1| = -3 < 0$$

$$B_2 = B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}, \quad |B_2| = 18 - 16 = 2 > 0$$

B is neg. definite

Homework: Find definiteness of the following symmetric matrices ต่อไปนี้

(a) $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

Second-Order Sufficient Condition (SOSC, SOC)For the maximum of objective function $z = f(x, y)$ the slope is decreasing ... $d^2z < 0$, d^2z neg def. $\Rightarrow H$ is neg def.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$|H_1| = f_{xx} < 0$$

$$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$$

For the minimum of objective function $z = f(x, y)$ the slope is increasing ... $d^2z > 0$, d^2z positive def. $\Rightarrow H$ is positive def.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$|H_1| = f_{xx} > 0$$

$$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 > 0$$

Example: Find the maximum/minimum of function $z = x + 2ey - e^x - e^{2y}$

$$\text{FOC: } \frac{\partial z}{\partial x} = 1 - e^x = 0 \Rightarrow x^* = 0$$

$$\frac{\partial z}{\partial y} = 2e - 2e^{2y} = 0 \Rightarrow e = e^{2y}$$

$$\text{note: } \frac{d}{dx} e^{kx} = ke^{kx} \quad 2y = 1 \Rightarrow y^* = \frac{1}{2}$$

$$\text{SOC: } f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial (1 - e^x)}{\partial x} = -e^x$$

$$f_{xy} = f_{yx} = \frac{\partial f_x}{\partial y} = \frac{\partial (1 - e^x)}{\partial y} = 0$$

$$f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial (2e - 2e^{2y})}{\partial y} = -4e^{2y}$$

$$H = \begin{bmatrix} -e^x & 0 \\ 0 & -4e^{2y} \end{bmatrix}$$

$$|H_1| = |-e^x| = -e^x < 0$$

$$|H_2| = 4e^{2+2y} = 4e^{0+2 \cdot \frac{1}{2}} = 4e > 0$$

H is neg def. $\therefore (x^*, y^*)$ gives z_{\max}

$$z_{\max} = 0 + e - 1 - e = -1 \quad \#$$



Objective functions with more three choice variables

$$z = f(x_1, x_2, x_3)$$

F.O.N.C :

$$dz = 0$$

$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0; dx_1, dx_2, dx_3 > 0$$

$$f_1 = f_2 = f_3 = 0$$

$$f_1 = f_2 = f_3 = \dots = f_n = 0$$

We can use $f_1 = f_2 = f_3 = 0$ to find the stationary values/critical values

S.O.S.C

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}_{3 \times 3}$$

All leading principal minor

$$|H_1| = |f_{11}|$$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

z is at the $\begin{bmatrix} \text{maximum} \\ \text{minimum} \end{bmatrix}$ if $\begin{cases} d^2z < 0 \\ d^2z > 0 \end{cases}$ if $\begin{cases} |H_1| < 0, |H_2| > 0, |H_3| < 0 \\ |H_1| > 0, |H_2| > 0, |H_3| > 0 \end{cases}$ if $\begin{cases} H \text{ is negative definite} \\ H \text{ is positive definite} \end{cases}$



Objective functions with more n –choice variables

$$z = f(x_1, x_2, x_3, \dots, x_n)$$

F.O.N.C :

$$dz = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n = 0$$

$$\frac{\partial^2 f_i}{\partial x_i^2} = 0, \text{ for all } i, i = 1, 2, \dots, n$$

S.O.S.C:

$$\therefore |H| = \begin{vmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix}$$

All principal minors $|H_1|, |H_2|, \dots, |H_n|$

If z is at is the maximum when $d^2z < 0$, H is negative definite:

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, |H_4| > 0, |H_5| < 0, \dots$$

If z is at is the minimum when $d^2z > 0$, H is positive definite:

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, |H_4| > 0, |H_5| > 0, \dots$$

Homework: Find extreme values of the following functions and check for the sufficient condition

$$z = 8x^3 + 2xy - 3x^2 + y^2 + 1$$

$$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$$

$$z = -3x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$$

$$z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$$

$$z = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$$

$$z = e^x + e^y + e^{w^2} - 2e^w - (x + y)$$

$$z = x^4 + x^2 - 6xy + 3y^2$$

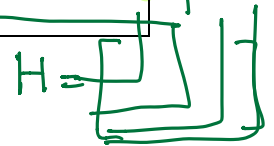
$$w = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$$

$$w = (x^2 + 2y^2 + 3z^2)e^{-(x^2 + y^2 + z^2)}$$

Summary of optimal conditions

The objective function: $z = f(x, y)$

Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient (SOSC, SOC)	$ H_1 = f_{xx} < 0$ $ H_2 = f_{xx}f_{yy} - f_{xy}^2 > 0$	$ H_1 = f_{xx} > 0$ $ H_2 = f_{xx}f_{yy} - f_{xy}^2 > 0$



The objective function: $z = f(x_1, x_2, \dots, x_n)$

Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	$f_1 = f_2 = \dots = f_n = 0$	$f_1 = f_2 = \dots = f_n = 0$
Second-order sufficient (SOSC, SOC)	$d^2z < 0$, negative def. H is negative definite. $(-1)^k H_k > 0$ $k = 1, 2, \dots, n$	$d^2z > 0$, positive def. H is positive definite. $ H_k > 0$ $k = 1, 2, \dots, n$

$(-1)^1 |H_1| > 0$
 $(-1)^2 |H_2| > 0$

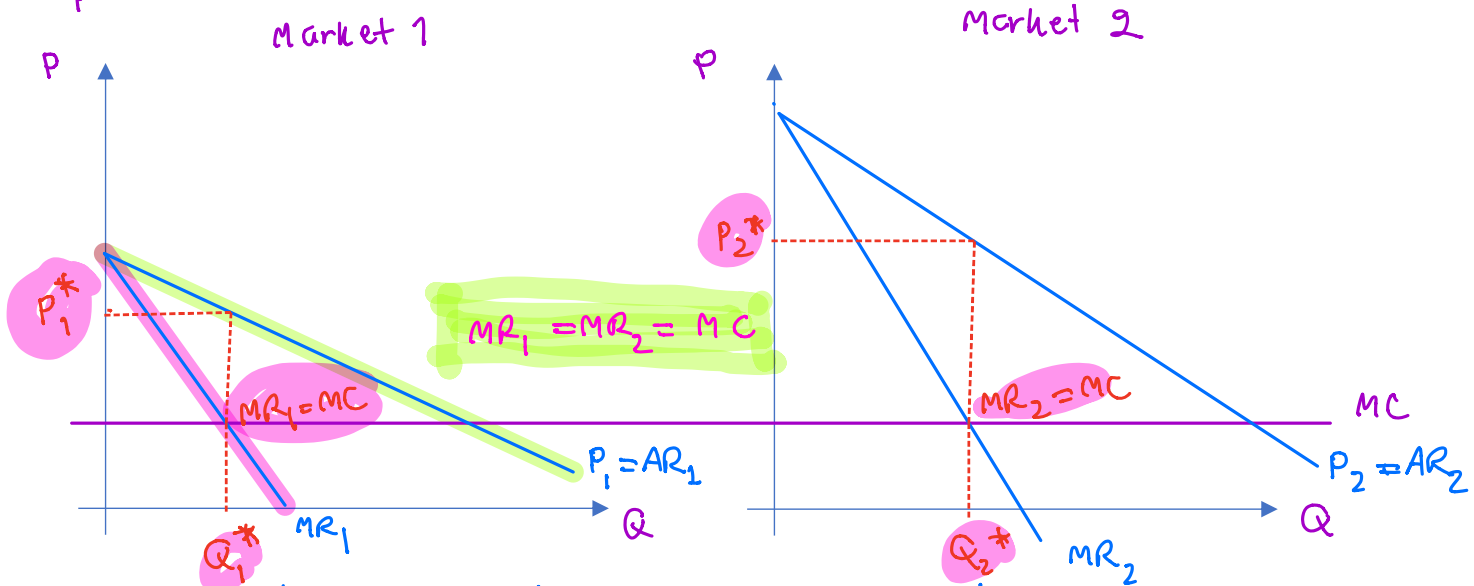


Third- Degree Price Discrimination

Monopolist

Third-Degree Price Discrimination involves charging different prices to different segments of the market for the same good. This is to maximize profit within each segment of the market. For example, a theater may divide cinema goes into seniors, adults, and children, each paying a different price when seeing the same movie.

Example : Assume $MC = \text{constant}$



Suppose that there are three segmented markets

Total Revenue of a company is:

$$TR = R_1(Q_1) + R_2(Q_2) + R_3(Q_3),$$

R_i is a total revenue function of market i .

segment market according to the nature of consumer's demand in each market

Total cost of the company is:

$$C = C(Q)$$

total quantity produced

$$Q = Q_1 + Q_2 + Q_3$$

Q_i is quantity of product sold in market i .

Find the quantity and price in each market i that maximizes firm's profit.

Step 1 State the objective Function: Profit function

$$\pi = TR - TC$$

$$\max_{Q_1, Q_2, Q_3} \pi = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q) ; Q = Q_1 + Q_2 + Q_3$$

$\max_{Q_1, Q_2, Q_3} \pi = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q)$; $Q = Q_1 + Q_2 + Q_3$

Step 2 Find FONC and use FONC to find critical values

FoC

$\frac{\partial \pi}{\partial Q_1} = \pi_1 = R_1'(Q_1) - C'(Q) \frac{\partial Q}{\partial Q_1} = 0 \quad (1)$
 $\frac{\partial \pi}{\partial Q_2} = \pi_2 = R_2'(Q_2) - C'(Q) \frac{\partial Q}{\partial Q_2} = 0 \quad (2)$
 $\frac{\partial \pi}{\partial Q_3} = \pi_3 = R_3'(Q_3) - C'(Q) \frac{\partial Q}{\partial Q_3} = 0 \quad (3)$

use these to derive Q_1^*, Q_2^*, Q_3^*
 $\downarrow \quad \downarrow \quad \downarrow$
 $P_1^* \quad P_2^* \quad P_3^*$
 (to get P^* the substitute Q^* into each segmented mkt. demand)

$C'(Q) = R_1'(Q_1) = R_2'(Q_2) = R_3'(Q_3)$

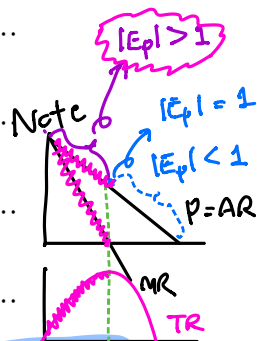
$MC = MR_1 = MR_2 = MR_3$

inverse mkt. demand of market i

From $R_i = P_i Q_i = P_i(Q_i) Q_i$

$MR_i = \frac{dR_i}{dQ_i} = P_i(Q_i) + Q_i \frac{dP_i(Q_i)}{dQ_i}$

$MR_i = P_i \left[1 + \frac{1}{E_{P_i}} \right]$



$MC = P_1 \left[1 + \frac{1}{E_{P_1}} \right] = P_2 \left[1 + \frac{1}{E_{P_2}} \right] = P_3 \left[1 + \frac{1}{E_{P_3}} \right]$

more less less more Mkt1 is less price elastic than Mkt 2

Conclusion: In the market with lower price elasticity of demand (lower $|E_{di}|$), the maximizing-profit price in that market will be more.

$|E_1^p| < |E_2^p|$
 $|-1.5| < |-2|$

Step 3 Check SOSC whether $Q_1^*, P_1^*, Q_2^*, P_2^*, Q_3^*, P_3^*$ indeed give the maximum profit

Hessian Matrix:

$H = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}$

$\frac{1}{E_p^1} < \frac{1}{E_p^2}$
 $-\frac{1}{1.5} < -\frac{1}{2}$
 $-0.67 < -0.5$

$1 + \frac{1}{E_p^1} < 1 + \frac{1}{E_p^2}$
 $1 - 0.67 < 1 - 0.5$

$\pi_i = MR_i - MC$
 $\pi_{ii} = \frac{\partial \pi_i}{\partial Q_i} = \frac{\partial (MR_i - MC)}{\partial Q_i} = \frac{\partial MR_i}{\partial Q_i} - \frac{\partial MC}{\partial Q_i}$
 $\pi_{ii} = \frac{\partial (P_i Q_i)}{\partial Q_i} - \frac{\partial MC}{\partial Q_i} = P_i + Q_i \frac{\partial P_i}{\partial Q_i} - \frac{\partial MC}{\partial Q_i}$

$$H = \begin{bmatrix} MR_1' - MC' & -MC' & -MC' \\ -MC' & MR_2' - MC' & -MC' \\ -MC' & -MC' & MR_3' - MC' \end{bmatrix}_{3 \times 3}$$

H_1 H_2 $H_3 = H$

Check all leading principal minors

H is neg def.

$$|H_1| = |\pi_{11}| = MR_1' - MC' < 0$$

$$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = (MR_1' - MC')(MR_2' - MC') - (MC')^2 > 0$$

$$|H_3| = \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{vmatrix} = \dots < 0$$

→ at $P_1^*, Q_1^*, P_2^*, Q_2^*, P_3^*, Q_3^*$, π is maximized.

Example: Suppose a monopolist firm sells its product in three markets with corresponding market demand:

Domestic market:

$$P_1 = 63 - 4Q_1$$

Asia market:

$$P_2 = 105 - 5Q_2$$

Europe market:

$$P_3 = 75 - 6Q_3$$

price = f(Q demanded)

Total cost function of this firm is:

$$C = 20 + 15Q; \\ Q = Q_1 + Q_2 + Q_3$$

Find total quantity produced by the monopolist and price and quantity in each market that maximize profit.

Step 1 State the objective Function: Profit function

$$\pi = TR - TC$$

$$= TR_1 + TR_2 + TR_3 - TC$$

$$= P_1 Q_1 + P_2 Q_2 + P_3 Q_3 - TC$$

$$\max_{Q_1, Q_2, Q_3} \pi = (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - 6Q_3)Q_3 - \{20 + 15Q\}; \quad Q = Q_1 + Q_2 + Q_3$$

$$\max_{Q_1, Q_2, Q_3} \pi = (63Q_1 - 4Q_1^2) + (105Q_2 - 5Q_2^2) + (75Q_3 - 6Q_3^2) - 20 - 15Q_1 - 15Q_2 - 15Q_3$$

Step 2 Find FONC and use FONC to find critical values

$$\text{FOC: } \frac{\partial \pi}{\partial Q_1} = 63 - 8Q_1 - 15 \stackrel{\text{FOC}}{=} 0 \quad (1.) \Rightarrow Q_1^* = 6$$

$$\frac{\partial \pi}{\partial Q_2} = 105 - 10Q_2 - 15 \stackrel{\text{FOC}}{=} 0 \quad (2.) \Rightarrow Q_2^* = 9$$

$$\frac{\partial \pi}{\partial Q_3} = 75 - 12Q_3 - 15 \stackrel{\text{FOC}}{=} 0 \quad (3.) \Rightarrow Q_3^* = 5$$

$$Q = 6 + 9 + 5 = 20$$

$$p_1^* = 63 - 4(6) = 39$$

$$p_2^* = 105 - 5(9) = 60$$

$$p_3^* = 75 - 6(5) = 45$$

Step 3: SOSC

$$H = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -12 \end{bmatrix}$$

Check all leading principal minors

$|H_1| = |-8| = -8$

< 0

$|H_2| = 80$

> 0

$|H_3| = -8 \times 10 \times -12 = -960$

< 0

$\therefore H$ is neg def. , Q_1^*, Q_2^*, Q_3^* maximize π
 P_1^*, P_2^*, P_3^* \oplus

Competitive Firm Input Choices: Cobb-Douglas Technology

Let a firm in competitive market have Cobb-Douglas Production Function:

$Q = L^\alpha K^\beta$

How much K, L should the firm employ in order to maximize its profit?

Let wage be w , rent be r baht per unit.

Characteristics of the production function:

Homogenous F^0 of degree $\alpha + \beta$

$MP_L = \frac{\partial Q}{\partial L} = \alpha L^{\alpha-1} K^\beta$

$MP_K = \frac{\partial Q}{\partial K} = \beta L^\alpha K^{\beta-1}$

Step 1 State the objective Function: Profit function

$\pi = TR - TC = PQ - (wL + rK)$

MAX $\pi = PL^\alpha K^\beta - wL - rK$
 K, L
 ↑ mkt price in an output mkt. ↑ mkt price in factor market

Step 2 Find FONC and use FONC to find critical values

FOC: $\pi_L = \frac{\partial \pi}{\partial L} = P\alpha L^{\alpha-1} K^\beta - w = 0$ (1)
 $\pi_K = \frac{\partial \pi}{\partial K} = P\beta L^\alpha K^{\beta-1} - r = 0$ (2) } solve for K^*, L^*

(1) $\Rightarrow P\alpha L^{\alpha-1} K^\beta = w$ — (3)

$P \times MP_L = w$

$MRP_L = VMP_L = w$

Marginal Revenue Product
 $MRP_L = MR \times MP_L$
 Value of marginal product
 $VMP_L = P \times MP_L$
 since $P = MR$
 in competitive mkt.
 $MRP_L = VMP_L$

$$(2.) \Rightarrow P \beta L^\alpha K^{\beta-1} = r \quad (4.)$$

$$P \times MP_K = r$$

$$MRP_K = VMP_K = r$$

$$(3.) / (4.) \quad \frac{P \alpha L^{\alpha-1} K^\beta}{P \beta L^\alpha K^{\beta-1}} = \frac{w}{r}$$

$$\frac{\alpha K}{\beta L} = \frac{w}{r}$$

$$* \left[\frac{K}{L} = \frac{\beta w}{\alpha r} \right] \quad (5.)$$

\Rightarrow the ratio of K to L to achieve maximum profit (if SOC is satisfied)

$$(5.) \Rightarrow K = \frac{\beta w}{\alpha r} L \quad (6.)$$

$$(6.) \& (3.) \quad P \alpha L^{\alpha-1} \left[\frac{\beta w}{\alpha r} L \right]^\beta = W$$

$$P \alpha^{1-\beta} \beta^\beta w^\beta r^{-\beta} L^{\alpha+\beta-1} = W$$

Later demand function: $L^* = \left[\frac{P \alpha^{1-\beta} \beta^\beta w^{\beta-1} r^{-\beta}}{W} \right]^{\frac{1}{1-\alpha-\beta}}$

$$(5.) \Rightarrow L = \frac{\alpha r}{\beta w} K \quad (7.)$$

wage \uparrow $L^* \downarrow$

(7.) $\&$ (4.)

or (a.) $0 < \beta < 1$ $\Rightarrow 0 < \alpha + \beta < 1$

(b.) $\beta > 1$ $\Rightarrow \alpha + \beta > 1$

for the law of demand to be satisfied

The Economic Interpretation

Capital demand function: $K^* = \left[\frac{P \alpha^{1-\beta} \beta^\beta w^{\beta-1} r^{-\beta}}{W} \right]^{\frac{1}{1-\alpha-\beta}}$

$$MRP_L = VMP_L = P \times MP_L = W$$

$$MRP_K = VMP_K = P \times MP_K = r$$

(a.) $0 < \alpha < 1$ $\Rightarrow 0 < \alpha + \beta < 1$

or (b.) $\alpha > 1$ $\Rightarrow \alpha + \beta > 1$

for the law of demand to be satisfied.

$$\pi_L = \frac{\partial \pi}{\partial L} = P\alpha L^{\alpha-1} K^\beta - W \Rightarrow H = \begin{bmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{bmatrix}$$

$$\pi_K = \frac{\partial \pi}{\partial K} = P\beta L^\alpha K^{\beta-1} - r$$

π_{KK}
4x4

Step 3: SOSC

$$H = \begin{bmatrix} P\alpha(\alpha-1)L^{\alpha-2}K^\beta & P\alpha\beta L^{\alpha-1}K^{\beta-1} \\ P\alpha\beta L^{\alpha-1}K^{\beta-1} & P\beta(\beta-1)L^\alpha K^{\beta-2} \end{bmatrix}$$

$\pi_{LK} = \pi_{KL}$

$0 < \alpha < 1$

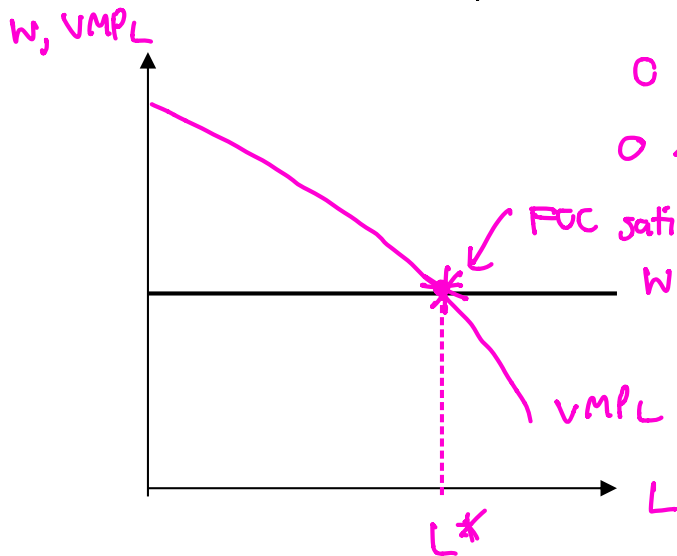
Check all leading principal minors

$|H_1| = P\alpha(\alpha-1)L^{\alpha-2}K^\beta < 0$

$|H_2| = P^2\alpha\beta(\alpha-1)(\beta-1)L^{2\alpha-2}K^{2\beta-2} - P^2\alpha^2\beta^2L^{2\alpha-2}K^{2\beta-2}$

} restrictions to be imposed so that H is neg def. So that at L^*, K^* , we have π_{max}

$0 < \alpha < 1$
 $0 < \beta < 1$





Multiproduct Firm

Multiproduct Firm in competitive market



e.g. a firm producing dish-washing liquid & shampoo

Let a firm produce and sell two products. Each product is sold in competitive market, hence the firm is price taker. Total Revenue is:

$$TR = P_1 Q_1 + P_2 Q_2$$

P_1 is market price of good 1.

P_2 is market price of good 2.

Q_1 is quantity of good 1 sold in the market.

Q_2 is quantity of good 2 sold in the market.

} firm is a price taker : $MR_1 = \frac{\partial TR}{\partial Q_1} = P_1$
 $MR_2 = \frac{\partial TR}{\partial Q_2} = P_2$

Total cost of this firm is:

$$C = 2Q_1^2 + Q_1 Q_2 + 2Q_2^2 \Rightarrow \text{Production of good 1 depends on good 2 \& vice versa.}$$

$$MC_1 = \frac{\partial C}{\partial Q_1} = 4Q_1 + Q_2 \Rightarrow MC_1 \text{ depends on good 2}$$

$$MC_2 = \frac{\partial C}{\partial Q_2} = Q_1 + 4Q_2 \Rightarrow MC_2 \text{ depends on good 1}$$

Firm's Profit Function

$$\pi = TR - TC$$

$$\max_{Q_1, Q_2} \pi = P_1 Q_1 + P_2 Q_2 - 2Q_1^2 - Q_1 Q_2 - 2Q_2^2$$

Choice Variables: Q_1, Q_2

Find FONC and use FONC to find critical values

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = P_1 - 4Q_1 - Q_2 = 0 \quad (1.)$$

$$MR_1 - MC_1 = 0 \Rightarrow MR_1 = MC_1 \quad (1.^*)$$

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = P_2 - Q_1 - 4Q_2 = 0 \quad (2.)$$

$$MR_2 - MC_2 = 0 \Rightarrow MR_2 = MC_2 \quad (2.^*)$$

$$\frac{(1.^*)}{(2.^*)} \quad \frac{MR_1}{MR_2} = \frac{MC_1}{MC_2}$$

$\frac{MR_1}{MC_1} = \frac{MR_2}{MC_2}$ | Bahat investka into each product must gain equal marginal benefit

Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$|H_1| =$$

$TR =$

.....

.....

Let the cost function of the monopolist be

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

Profit Function:

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Choice Variables:

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$|H_1| =$$

$$|H_2| =$$

The second order differential of π , $d^2\pi$, isdefinite. Hence, the profit is.....

.....



Multiplant Firm Problem

Consider a firm with many plants. Each plant i has total cost TC_i .

$$TC_i = C_i(q_i), i = 1, 2, \dots, n$$

q_i is the level of output produced by factory i .

Assume that this firm sells its product in one market. Firm's total revenue function is:

$$TR = \dots\dots\dots$$

$$Q = \dots\dots\dots$$

If output market is competitive market,

.....

If output market is monopoly market,

.....

Firm's Profit Function

$$\pi = TR - TC$$

.....

Choice Variables:

NOTE: Third-degree price discrimination vs. Multiplant firm

.....

First Order Necessary Condition:

$$\pi_i = \dots \text{for all } i = 1, 2, \dots, n$$

SOSC:

$$H = \left[\begin{array}{c} \\ \\ \end{array} \right]$$

$$|H_1| = R'' - C_1'' < 0$$

$$|H_2| =$$

Economic interpretation:

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Example: Consider a monopolist firm with two plants:

Factory 1: $C_1(Q_1) = 10Q_1^2$

Factory 2: $C_2(Q_2) = 20Q_2^2$

.....

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Firm's market demand is $P = 700 - 5Q$, $Q = Q_1 + Q_2$

Draw MC_1, MC_2, MC_T, AR, MR and indicate the maximizing level of output from each plant and the profit-maximizing price.



Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$|H_1| =$$

$$|H_2| =$$

The second order differential of π , $d^2\pi$, isdefinite. Hence, the profit is.....

.....

Summary:

3rd price discrimination:

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Choosing levels of factor of production:

.....

Multiproduct:

.....

Multiplant:

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