

## CHAPTER 4

## Basic Matrix Algebra and Applications

Complete  
version

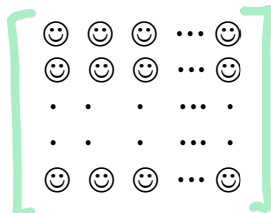
Topics: **Basic Matrix Algebra**

Outline:

- ☺ Representation of system of equation by matrix notation
- ☺ Multiplication of matrices
- ☺ Determinant and singularity of matrix
- ☺ Matrix Inversion
- ☺ Cramer's rule

## Matrix Algebra

**Matrix** is a rectangular array of numbers, parameters, or variables



$$\begin{matrix}
 \text{row } 1 \\
 \text{row } m
 \end{matrix}
 \left[ \begin{array}{ccc}
 a_{11} & a_{12} & \dots & a_{1n} \\
 a_{21} & a_{22} & \dots & a_{2n} \\
 \vdots & \vdots & \dots & \vdots \\
 a_{m1} & a_{m2} & \dots & a_{mn}
 \end{array} \right]_{m \times n} = \text{dimension of this matrix}$$

$$[a_{ij}]_{m \times n}, i = 1, \dots, m; j = 1, \dots, n$$

### Benefits of using Matrix

1. Matrix provides a compact way of writing an equation system, even an extremely large one
2. Matrix provides a way of testing the existence of a solution by evaluation of a determinant.
3. Matrix can be used to find that solution if exists. This is done by using Inverse matrix or Cramer's rule

*Limitation:* matrix algebra is applicable only to linear-equation systems.

In some cases, we can transform variables so as to obtain a linear relation to work with. For example, the nonlinear function  $y = ax^b$  can be transformed into:

$$\log(y) = \log(a) + b \log(x)$$



**Definition 2**

Given a matrix  $A = [a_{ij}] \in \mathbb{R}^{m \times n}$  and a scalar  $c \in \mathbb{R}$  the **scalar multiplication of matrix A** by the scalar  $c$  is given by  $cA = [ca_{ij}] \in \mathbb{R}^{m \times n}$

**Example**

$$7 \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} =$$

**Definition 3**

The **additional of two matrices**  $A = [a_{ij}] \in \mathbb{R}^{m \times n}$  and  $B = [b_{ij}] \in \mathbb{R}^{p \times q}$  is defined only if  $m=p$  and  $n=q$ , and is given by

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \in \mathbb{R}^{m \times n}$$

**Example**

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} =$$

**Transposition of a Matrix****Definition 6**

Let  $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ , The transpose of  $A$  is given by,

$$A^T = [a_{ij}^T] \in \mathbb{R}^{n \times m}, \text{ where } a_{ij}^T = a_{ji}$$

When a matrix is transposed, the rows becomes columns and vice versa

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2 \times 3}$$

$$A^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}_{3 \times 2}$$

**Example**

$$A = \begin{bmatrix} 3 & 8 & 9 \\ 1 & 0 & 4 \end{bmatrix} \quad A' =$$

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} \quad B' =$$

$$(AB)^T = B^T A^T$$

### Recapping some special types of matrices

>> Square matrix

A matrix  $A = [a_{ij}] \in \mathbb{R}^{m \times n}$  is a **square matrix** if  $m=n$ . That is, the number of rows is equal to the number of columns

**Example**

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 5 & 2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

>> Diagonal matrix

The **diagonal elements** of a square matrix

$A \in \mathbb{R}^{m \times n}$  are the elements

$$a_{11}, a_{22}, \dots, a_{mm}$$

A **diagonal matrix** is a square matrix

$$A = [a_{ij}] \in \mathbb{R}^{n \times n}$$

where  $a_{ij} = 0$  if  $i \neq j$

$$\text{with } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

>> Identity matrix

An **identity matrix** is a diagonal matrix  $I_n$  of order  $n \times n$  whose diagonal elements being ones. An identity matrix is thus necessarily square.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

>> Null matrix

A matrix  $A = [a_{ij}] \in \mathbb{R}^{m \times n}$  is a **zero matrix/null matrix** if all of its elements are zeros. We write

$$A = \mathbf{0}$$

**Example**

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

>>Triangular matrix

A matrix “A” is **upper triangular** if it is square and  $a_{ij} = 0$  if  $i > j$ , and **lower triangular** if it is square and  $a_{ij} = 0$  if  $i < j$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 6 & 5 & 4 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

>> Symmetric matrix

A matrix is symmetric if  $A^T = A$ . That is  $a_{ij} = a_{ji}$  for all  $i, j$  in  $n$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

**We will learn the following matrix operation (Matrix Algebra)**

↗ Multiplication of matrices

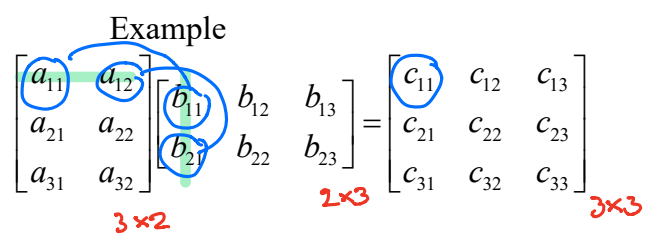
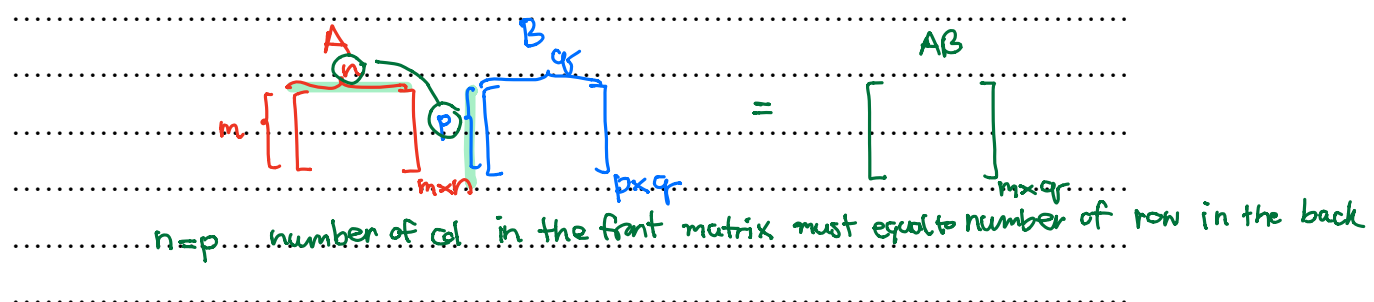
↗ Determinant of a matrix

↗ Inverses

**↗ Multiplication of matrices**

The Multiplication of matrix  $A = [a_{ij}]_{m \times n}$  by matrix  $B = [b_{ij}]_{p \times q}$  is defined only if

$n = p$  and is given by  $AB = \left[ \sum_{k=1}^n a_{ik} b_{kj} \right]_{m \times q}$



$c_{11} = a_{11}b_{11} + a_{12}b_{21}$   
 from row 1 of the front  
 k col 1 of the back

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{33} = a_{31}b_{13} + a_{32}b_{23}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix}_{2 \times 2} = \begin{bmatrix} 19 & 22 \\ 43 & 47 \end{bmatrix}_{2 \times 2}$$

Question:  $A_{4 \times 3} B_{2 \times 4} = ?$

This multiplication is not possible because  $n \neq p$

“dot product”

### Definition 5

When a row vector is multiplied by a column vector, the product is called **dot product or inner product**. That is, if

$$\mathbf{a} \mathbf{b} = [a_1 \quad a_2 \quad \dots \quad a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \left[ \sum_{i=1}^n a_i b_i \right] \in \mathbf{R}^{1 \times 1} = \mathbf{R},$$

$$u = [3 \quad 4]_{1 \times 2} \quad v = \begin{bmatrix} 9 \\ 7 \end{bmatrix}_{2 \times 1} \quad uv = [3 \cdot 9 + 4 \cdot 7] = [55] = 55$$

Note: A matrix is idempotent if  $A^2 = A$

$$\mathbf{A} \mathbf{A} = \mathbf{A}$$

$$A = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} \quad \mathbf{A} \mathbf{A} = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} = \mathbf{A}$$

↗ **Determinant** → to check if there is solution to the system of equation  
 → use in finding the inverse

The determinant of a square matrix  $A$ , denoted by  $|A|$ , is a **uniquely defined number** associated with that matrix

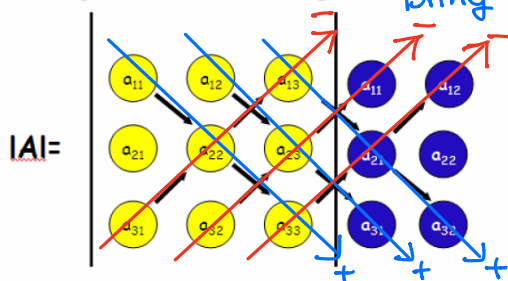
For 2 by 2 matrix  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ , the 2<sup>nd</sup> order determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \dots a_{11} \cdot a_{22} - a_{21} \cdot a_{12} \dots$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

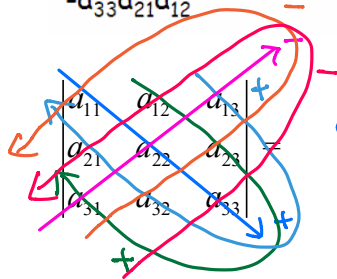
### Evaluating a Third-Order Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ its determinant is}$$



Bring the first two columns to the back

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$



$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

$n^{\text{th}}$ -order determinant by Laplace Expansion

Let  $A$  be a matrix in  $R^{n \times n}$ . The determinant of  $A$  or  $\det A$  is given by  $|A|$ .

$$|A| = \begin{cases} \sum_{j=1}^n a_{ij} C_{ij} & i = 1, 2, \dots, n \text{ (expansion by } i^{\text{th}} \text{ row)} \\ & \text{sum across column} \\ \sum_{i=1}^n a_{ij} C_{ij} & j = 1, 2, \dots, n \text{ (expansion by } j^{\text{th}} \text{ column)} \\ & \text{sum across row} \end{cases}$$

If we choose row  $i$ ,

$$\begin{bmatrix} a_{11} & a_{12} & a_{j1} & a_{1n} \\ a_{21} & a_{22} & a_{j2} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n \times n}$$

$$|A| = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{ij} C_{ij} + \dots + a_{in} C_{in}$$

If we choose column  $j$ ,

$$\begin{bmatrix} a_{11} & a_{12} & a_{j1} & a_{1n} \\ a_{21} & a_{22} & a_{j2} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n \times n}$$

$$|A| = a_{j1} C_{j1} + a_{j2} C_{j2} + \dots + a_{ij} C_{ij} + \dots + a_{nj} C_{nj}$$

>> We call  $C_{ij}$ , Cofactor of element  $a_{ij}$ .

any row, any column  
will give the same det.

>>  $C_{ij}$  can be found by  $(-1)^{i+j} M_{ij}$

The **minor of element  $a_{ij}$ ,  $M_{ij}$** , is the determinant of the matrix resulting from deleting row  $i$  and column  $j$  of matrix  $A_{n \times n}$ . In other words, a "minor" is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21} a_{33} - a_{23} a_{31}$$

**Tip** choose row or Column with many 0

Example, find the following determinant:

$$\begin{aligned}
 \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} &= 7C_{12} + 5C_{22} + 0C_{32} \\
 &= 7(-1)^{1+2} \begin{vmatrix} 2 & 6 \\ 9 & 12 \end{vmatrix} + 5(-1)^{2+2} \begin{vmatrix} 15 & 9 \\ 9 & 12 \end{vmatrix} \\
 &= -7(24 - 54) + 5(180 - 81) \\
 &= 705
 \end{aligned}$$

$$\begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} = 15(5)(12) + 7(6)(9) + 9(2)(0) - 9(5)(9) - 0(6)(15) - 12(2)(7) = 705$$

## Properties of Determinants

### Property I

The interchange of rows and columns does not affect the value of a determinant. In other words, the determinant of a matrix  $A$  has the same value as that of its transpose  $A'$ , that is

$$\begin{aligned}
 \det A &= \det A^T \\
 |A| &= |A^T|
 \end{aligned}$$

### Example

$$A = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix} \quad \det A = 24 - 15 = 9$$

$$B = \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix} = A' \quad \det B = 24 - 15 = 9$$

**Property II**

The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value, of the determinant

**Example**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = ad - bc$$

$$B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\det B = bc - ad$$

**Property III**

The multiplication of any one row (or one column) by a scalar  $k$  will change the value of the determinant  $k$ -fold

**Example**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = ad - bc$$

$$B = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

$$\det B = kad - kbc = k(ad - bc) = k \det A$$

**Property IV**

The addition (subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered

**Example**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det A = ad - bc$$

$$B = \begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix} \quad \det B = ad + kab - bc - kab = ad - bc$$

$$r_{2,B} = kr_{1,A} + r_{2,A}$$

**Property V**

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero

Example

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det A =$$

$$B = \begin{pmatrix} 2a & 2b \\ a & b \end{pmatrix} \quad \det B = 2ab - 2ab = 0$$

$r_1 = 2r_2$

**The determinant & Non-singularity**

If  $|A| = 0$ , square matrix  $A_{n \times n}$  is a singular Matrix

: A square matrix that is not invertible is called **singular** or **degenerate**. A square matrix is singular if and only if its determinant is 0

We cannot find  $A^{-1}$

If  $|A| \neq 0$ , matrix A is a **non-singular Matrix**

$A^{-1}$  exists, we can find  $A^{-1}$

↗ The inverse of A

Assume that an  $n \times n$  nonsingular matrix A is given by:

$$\begin{bmatrix} a_{11} & a_{12} & a_{j1} & a_{1n} \\ a_{21} & a_{22} & a_{j2} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n \times n}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}A$$

$[C_{ij}]^T$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

deleting row  $i$ , col  $j$

$$\begin{bmatrix} a_{11} & a_{12} & a_{j1} & a_{1n} \\ a_{21} & a_{22} & a_{j2} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{bmatrix}_{n \times n}$$

To find the inverse of matrix  $A$ :

- 1.) Find  $\det A, |A|$ , if  $|A| = 0$ ,  $A$  is a singular matrix and the inverse of matrix  $A$  cannot be found.
- 2.) Find Minor ( $M_{ij}$ ) for all element of the matrix in row  $i = 1, \dots, n$  and column  $j = 1, \dots, n$
- 3.) Find Cofactor ( $C_{ij}$ ) from Minor ( $M_{ij}$ ) for all elements of the matrix in row  $i = 1, \dots, n$  and column  $j = 1, \dots, n$
- 4.) Get the Cofactor matrix  $C$  in which each element is Cofactor ( $C_{ij}$ ) of element  $a_{ij}$
- 5.) The transpose of  $C$  is called Adjoint  $A$  ( $\text{adj}A$ )

$$\text{adj}A = [C_{ij}]^T$$

- 6.) Multiply  $\text{adj}A$  by scalar  $\frac{1}{\det A}$   $\text{adj}A$  and we will get the inverse of matrix  $A$ ,  $A^{-1}$ .

Properties of the inverse matrix  $A^{-1}$

$$\begin{aligned} \text{I.} & \dots (A^{-1})^{-1} = A \\ \text{II.} & \dots (AB)^{-1} = B^{-1}A^{-1} \\ \text{III.} & \dots (A^T)^{-1} = (A^{-1})^T \\ \text{IV.} & \dots A^{-1}A = AA^{-1} = I \end{aligned}$$

Find  $A^{-1}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = ad - bc \neq 0 \quad \dots \rightarrow ad \neq bc$$

$$M_{11} = d \quad M_{12} = c$$

$$M_{21} = b \quad M_{22} = a$$

$$C_{11} = (-1)^{1+1}d = d \quad C_{12} = (-1)^{1+2}c = -c$$

$$C_{21} = (-1)^{2+1}b = -b \quad C_{22} = (-1)^{2+2}a = a$$

$$\therefore \text{cof}A = C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\text{Adj}A = \text{cof}A^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{-2} \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/2 & -3/2 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 0 & 1 \\ 1/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A = \begin{pmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{pmatrix} \quad \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} = 15(5)(12) + 7(6)(9) + 9(2)(0) \\ - 9(5)(9) - 0(6)(15) - 12(2)(7) \\ = 705 \neq 0 \quad \text{non singular matrix}$$

$$\text{cof}A = \begin{matrix} 15 & 7 & 9 & 15 & 7 & 9 & 15 & 7 & 9 \\ 2 & 5 & 6 & 2 & 5 & 6 & 2 & 5 & 6 \\ 9 & 0 & 12 & 9 & 0 & 12 & 9 & 0 & 12 \\ 15 & 7 & 9 & 15 & 7 & 9 & 15 & 7 & 9 \\ 2 & 5 & 6 & 2 & 5 & 6 & 2 & 5 & 6 \\ 9 & 0 & 12 & 9 & 0 & 12 & 9 & 0 & 12 \\ 15 & 7 & 9 & 15 & 7 & 9 & 15 & 7 & 9 \\ 2 & 5 & 6 & 2 & 5 & 6 & 2 & 5 & 6 \\ 9 & 0 & 12 & 9 & 0 & 12 & 9 & 0 & 12 \end{matrix} \quad \text{cof}A = \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \begin{matrix} \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} & \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} & \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} \\ \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} & \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} & \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} \\ \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} & \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} & \begin{vmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} \end{matrix}$$

$$\text{cof}A = \begin{bmatrix} 60 & 30 & -45 \\ -84 & 99 & 63 \\ -3 & -72 & 61 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \text{adj}A = \frac{1}{705} \begin{bmatrix} 60 & -84 & -3 \\ 30 & 99 & -72 \\ -45 & 63 & 61 \end{bmatrix}$$

**To find the solution for system of equation by using matrix**

We can use (1.) Inverse matrix (2.) Cramer's rule  
*n variables*

Remind that:

$$\begin{array}{l}
 n \text{ eqs.} \\
 \left[ \begin{array}{l}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = d_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = d_2 \\
 \vdots \\
 a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = d_n
 \end{array} \right.
 \end{array}
 \quad \begin{array}{l}
 \text{row } i \Rightarrow \text{eq } i \\
 \text{col } j \Rightarrow \text{var } j
 \end{array}$$

We can write this system of equations in the form of matrix as:

$$\begin{bmatrix}
 a_{11} & a_{12} & \dots & a_{1n} \\
 a_{21} & a_{22} & \dots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n1} & a_{n2} & \dots & a_{nn}
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_n
 \end{bmatrix}
 =
 \begin{bmatrix}
 d_1 \\
 d_2 \\
 \vdots \\
 d_n
 \end{bmatrix}$$

$n \times n \quad n \times 1 \quad n \times 1$

$$\begin{array}{ccc}
 A & x & = & d \\
 n \times n & n \times 1 & & n \times 1
 \end{array}$$

To find solution to linear system of equations: $A_{n \times n} x_{n \times 1} = d_{n \times 1}$ CHECK IF DETERMINANT OF MATRIX A IS NOT EQUAL TO ZERO.			
Vector d		$d_{n \times 1} \neq 0$	$d_{n \times 1} = 0$ (homogenous equation system)
Det A			
$ A  \neq 0$ (Linear independent eq) Matrix A is non-singular $A^{-1}$ exist		Unique solution $x = A^{-1}d \neq \underline{0}$ e.g. $\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 8 \end{bmatrix}$	Unique solution $x = A^{-1}\underline{0} = \underline{0}$ e.g. $\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$ A  = 0$ Matrix A singular $A^{-1}$ does not exist	Linear dependent eq. $ A_k  = 0,  A  = 0$	Infinite solutions (not including $x = \underline{0}$ ) e.g. $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$	Infinite solutions (including $x = \underline{0}$ ) e.g. $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
	Inconsistent eq. $ A_k  \neq 0,  A  = 0$	No solution $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 65 \end{bmatrix}$	n/a

**Exercise:** Consider the following system of equations and examine whether there is(are) solution(s) to the system or not.

- |     |                   |     |                     |
|-----|-------------------|-----|---------------------|
|     | $4x + y + 2z = 5$ |     | $3x = 12 - 5y + 4z$ |
| 1.) | $5x + 2y + z = 7$ | 2.) | $6x = 9 + 10y$      |
|     | $x + z = 1$       |     | $8x = 10 + 7y - 3z$ |

>> Use inverse matrix

$$Ax = d$$

$$A^{-1}Ax = A^{-1}d$$

$$x_{n \times 1} = A^{-1}_{n \times n} d_{n \times 1}$$

>> Use Cramer's Rule:

From,

$$X_{n \times 1} = A_{n \times n}^{-1} d_{n \times 1}$$

That is,

$$X = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$; C_{ij} = (-1)^{i+j} M_{ij} \quad \text{and}$$

$$|A| = \begin{cases} \sum_{j=1}^n a_{ij} C_{ij} & i = 1, 2, \dots, n \text{ (expansion by } i^{\text{th}} \text{ row)} \\ \sum_{i=1}^n a_{ij} C_{ij} & j = 1, 2, \dots, n \text{ (expansion by } j^{\text{th}} \text{ column)} \end{cases}$$

$$X = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$X = \begin{pmatrix} \frac{1}{|A|} (C_{11}d_1 + C_{21}d_2 + \cdots + C_{n1}d_n) \\ \frac{1}{|A|} (C_{12}d_1 + C_{22}d_2 + \cdots + C_{n2}d_n) \\ \vdots \\ \frac{1}{|A|} (C_{1n}d_1 + C_{2n}d_2 + \cdots + C_{nn}d_n) \end{pmatrix} = \begin{pmatrix} \frac{1}{|A|} \sum_{i=1}^n C_{i1}d_i \\ \frac{1}{|A|} \sum_{i=1}^n C_{i2}d_i \\ \vdots \\ \frac{1}{|A|} \sum_{i=1}^n C_{in}d_i \end{pmatrix}$$

$C_{11}d_1 + C_{21}d_2 + \dots + C_{n1}d_n$  is the determinant of matrix A when replacing column 1 of matrix A by vector d. That is,

$$C_{11}d_1 + C_{21}d_2 + \dots + C_{n1}d_n = \begin{vmatrix} d_1 & a_{12} & \dots & a_{1n} \\ d_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_n & a_{n2} & \dots & a_{nn} \end{vmatrix} = |A_1|$$

do laplace expansion over col 1  
= det of matrix A when col 1 is replaced by vector d

Likewise,

$$\sum_{i=1}^n C_{i2}d_i = \begin{vmatrix} a_{11} & d_1 & \dots & a_{1n} \\ a_{21} & d_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & d_n & \dots & a_{nn} \end{vmatrix} = |A_2|$$

= det of matrix A when col 2 is replaced by vector d

$$\sum_{i=1}^n C_{in}d_i = \begin{vmatrix} a_{11} & a_{12} & \dots & d_1 \\ a_{21} & a_{22} & \dots & d_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & d_n \end{vmatrix} = |A_n|$$

= det of matrix A when col n is replaced by vector d

Therefore, Cramer's rule is

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{|A_1|}{|A|} \\ \frac{|A_2|}{|A|} \\ \vdots \\ \frac{|A_n|}{|A|} \end{bmatrix}$$

$x_k = \frac{|A_k|}{|A|}$

Example,

$$\text{Eq.1 } 5x + 3y = 30$$

$$\text{Eq.2 } 6x - 2y = 8$$

Find the solution to this system of equations by (1.) Inverse matrix (2.) Cramer's rule

$$(1) \text{ Inverse matrix } \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 30 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-10 - 18} \begin{bmatrix} -2 & -3 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} 30 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{-2(30) - 3(8)}{-28} \\ \frac{-6(30) + 5(8)}{-28} \end{bmatrix} = \begin{bmatrix} \frac{-60 - 24}{-28} \\ \frac{-180 + 40}{-28} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$(2) \quad x = \frac{\begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix}} = \frac{-60 - 24}{-28} = 3$$

$$y = \frac{\begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix}} = \frac{40 - 180}{-28} = 5$$