Quiz 2

1. Let $D = \{-1, 0, 1\}$ and $E = \{0, 1, 2\}$. Consider the statement

$$\forall x \in D, \exists y \in E \text{ such that } (xy \leq 0) \to (x \leq 0) \wedge (y < 0).$$

(a) Write the negation for the above statement (without using negation symbol "$\sim$" in the final answer).

(b) Determine the truth value of the above statement. Explain your answer.

Answer: First, using the order of the logical operations: we have $p \to q \wedge r \equiv p \to (q \wedge r)$.
(a) Negation:

$$\begin{aligned}
& \sim (\forall x \in D, \exists y \in E \text{ such that } (xy \leq 0) \to (x \leq 0) \wedge (y < 0)) \\
\equiv\; & \exists x \in D \sim (\exists y \in E \text{ such that } (xy \leq 0) \to (x \leq 0) \wedge (y < 0)) \\
\equiv\; & \exists x \in D, \forall y \in E \text{ such that } \sim ((xy \leq 0) \to (x \leq 0) \wedge (y < 0)) \\
\equiv\; & \exists x \in D, \forall y \in E \text{ such that } (xy \leq 0) \wedge \sim ((x \leq 0) \wedge (y < 0)) \\
\equiv\; & \exists x \in D, \forall y \in E \text{ such that } (xy \leq 0) \wedge (\sim (x \leq 0) \vee \sim (y < 0)) \\
\equiv\; & \exists x \in D, \forall y \in E \text{ such that } (xy \leq 0) \wedge ((x > 0) \vee (y \geq 0))
\end{aligned}$$

(b) This statement is **false** because there is $x \in D$ such that the statement is false for all $y \in E$. One counterexample here is when $x = -1$:

(I) $x = -1$: The statement is always **false** for all $y = 0, 1, 2 \in E$.
- For $y = 0$: $(xy \leq 0)$ is true and $(x \leq 0) \wedge (y < 0)$ is false, i.e. $T \to F$.
- For $y = 1$: $(xy \leq 0)$ is true and $(x \leq 0) \wedge (y < 0)$ is false, i.e.$T \to F$.
- For $y = 2$: $(xy \leq 0)$ is true and $(x \leq 0) \wedge (y < 0)$ is false, i.e. $T \to F$.

Another counterexample is when $x = 0$:

(II) $x = 0$: the statement is false for all $y \in E$.
- For $y = 0$: $(xy \leq 0)$ is true and $(x \leq 0) \wedge (y < 0)$ is false, i.e. $T \to F$.
- For $y = 1$: $(xy \leq 0)$ is true and $(x \leq 0) \wedge (y < 0)$ is false, i.e.$T \to F$.
- For $y = 2$: $(xy \leq 0)$ is true and $(x \leq 0) \wedge (y < 0)$ is false, i.e. $T \to F$.

Note, using *any of the above: case(I)* $x = -1$ *or case(II)* $x = 0$ would be enough to show that the truth value of the statement is false (**don't have to use both**).

[Note also that when $x = 1$, the hypothesis $(xy \leq 0)$ is always false for all $y = 0, 1, 2 \in E$ and therefore the statement is true for all values of $y = 0, 1, 2$. So, we cannot use $x = 1$ as a counterexample.]