Solving Absolute-Value Inequalities

TU152: Fundamental Mathematics

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Absolute Value

Definition (Absolute Value)

For any real number x, the absolute value of x, denoted $|x|$, is defined as follows:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

From the definition, the absolute value $|x|$ can be viewed as "the distance of x from zero" on the number line. E.g.

- "Equality": Both -3 and 3 are three units from zero. This implies $|-3| = |3| = 3$.
- " Less than": The solution to $|x| < 3$ consists of the points that are less than three units away from zero, i.e.

  $$-3 < x < 3.$$

  The solution set to $|x| \leq 3$ includes the solutions to $|x| < 3$ and $x = -3, 3$.
- " Greater than": The solution to $|x| > 3$ consists of the points that satisfy either one of the following two inequalities

  $$x < -3 \quad \text{or} \quad x > 3.$$

  The solution set to $|x| \geq 3$ includes the solutions to $|x| > 3$ and $x = -3, 3$.

Properties of Absolute-Value Inequities

Theorem 1: Properties of Absolute-Value Inequities

Let x be a real number and a be a positive real number.

(1) $|x| < a$ if and only if $-a < x < a$.

(2) $|x| \leq a$ if and only if $-a \leq x \leq a$.

(3) $|x| > a$ if and only if $x < -a$ or $x > a$.

(4) $|x| \geq a$ if and only if $x \leq -a$ or $x \geq a$.

Example Find the solution set for each of the inequalities.

- $|x + 1| \leq -5$
- $|2x - 3| > -5$

Example Find the solution set for each of the inequalities.

- $|x + 1| \leq 5$
- $|2x - 3| > 5$

Properties of Absolute-Value Inequities

Theorem 2: Properties of Absolute-Value Inequities

Let x and y be real numbers.

(1) If $x \neq 0$, then $|x| > 0$.

(2) $|x| = 0$ if and only $x = 0$

(3) $|-x| = |x|, \quad |x| \geq |x|, \quad |x| \leq |x|$

(4) $-|x| \leq x \leq |x|$

(5) $|x + y| \leq |x| + |y|$

(6) $|x - y| \leq |x| + |y|$

(7) $||x| - |y|| \leq |x - y|$

(8) $|xy| \leq |x||y|$

(9) $\left|\frac{x}{y}\right| \leq \frac{|x|}{|y|}$

(10) $|x| < |y|$ if and only if $x^2 < y^2$

$|x| \leq |y|$ if and only if $x^2 \leq y^2$

Example:

Let x and y be real numbers. Prove that if $|x| < 3$ and $y > 0$, $|x + y| < y + 3$.

Example: Let x be any real number with $|x| < 3$. Prove that $|x^2 - 1| < 4|x + 1|$.

Example: Let x and y be positive real numbers. Show that if $|x+1| < x+y$, then

$$1 - y \; < \; \frac{y^2-1}{2x}.$$

Solving Absolute-Value Inequalities

Example: Find the solution set for the inequality:

$$2 < |2x - 1| \leq 5.$$

Example: Find the solution set for the inequality:

$$|x + 2| \leq |2x + 3|.$$

Example: Find the solution set for the inequality:

$$|x + 5| \geq 3 + |x|.$$

Example: Find the solution set for following inequality.

$$|x^2 + x - 6| < 6$$

Example: Find the solution set for following inequality.

$$\frac{x^2}{|x|-1} \leq |x+1|$$