BACHELOR of ECONOMICS
International Program

Thammasat University | Thailand

FN 312 Investments

Option Valuation: The binomial model

Road Map/key Ideas

- Determinants of option price
- How to price options using a binomial model

3

Option pricing

Some usage of options

- Trade on views of directions of the underlying asset or volatility
- Hedging risks
- Price assets that have embedded options such as the callable bonds. Callable bonds can be priced as a vanilla bond - a call option.
- Price “real option”. Future investment opportunity of the firm can be viewed as the firm having an option to invest in the future

4

Determinants of Option Prices

	call	put
• Stock price		
• Strike price		
• Volatility		
• Time to maturity		
• Interest rate		
• Dividends		

5

Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price		
• Volatility		
• Time to maturity		
• Interest rate		
• Dividends		

6

Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility		
• Time to maturity		
• Interest rate		
• Dividends		

7

Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility	+	+
• Time to maturity		
• Interest rate		
• Dividends		

8

Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility	+	+
• Time to maturity	+	+
• Interest rate		
• Dividends		

9

Determinants of Option Prices

	call	put
• Stock price	+	-
• Strike price	-	+
• Volatility	+	+
• Time to maturity	+	+
• Interest rate	+	-
• Dividends	-	+

10

Bounds on option price

- Consider 2 portfolios:1. Long a call option, and 2. long the stock and borrow PV(X+D). D is the dividend of the stock.
- At expiration of the option each portfolio pay off is

1. Call = Max (S_T-X,0)
2. S_T + D – (X+D) = S_T –X

- Therefore, C_t >= S_t –PV(X+D)
- And S_t >=C_t >= S_t –PV(X+D)

11

Bounds on option price

- Consider 2 portfolios:1. Long a put option, and 2. short sell the stock and lend PV(X+D). D is the dividend of the stock.
- At expiration of the option each portfolio pay off is

1. Put = Max (X-S_T,0)
2. X+D-(S_T + D) = X-S_T

- Therefore, P_t >= PV(X+D)-S_t
- And X >=P_t >= Max(PV(X+D)-S_t, 0)

12

Option pricing: binomial tree

- An option pricing model assumes a distribution of how the underlying asset (stock) evolves in the future.
- The binomial model assumes the binomial distribution.
- Key innovation: The payoff of an option can be replicated dynamically by a portfolio that consists of some position in the underlying stock and borrowing/lending.

Option pricing: binomial tree

Simple example: A call option expiring in 1 year with a strike price or $150. The risk-free rate is 10%

Suppose S_0 = $100 & S_1 = $\begin{cases} \$200 & 0.5 \\ \$\,50 & 0.5 \end{cases}$

What is the pay off of the option at maturity?

	Prob.	S_1	C_1
S_0	0.5	200	50
	0.5	50	0

Option pricing: binomial tree

What if we form a portfolio that invests in Δ shares the underlying stock and borrow some money?

The portfolio is $\Delta S_0 - D = C_0$ where D is a money market account (borrowing/lending)

pay off

$1/3S_0$-\$15.15

- 1/3 * 200 - \$15.15*1.1 = 66.67-16.17 =50
- 1/3 * 50 - \$15.15*1.1 = 16.67-16.17 = 0

This is the same as pay off from the call option!

So, $1/3S_0 - \$15.15 = C_0$

15

Option pricing: binomial tree

Form a portfolio of Δ shares of stock and short one call option.

ΔS_0-C

- Δ(200) - \$50
- Δ(50) - \$0

Select the "hedge ratio" , Δ, so that the payoff of the portfolio is the same in either state of the world:

Find Δ:	Δ(200) - 50	=	Δ(50)
	Δ(150)	=	50
	Δ	=	(1/3)

16

Option pricing: binomial tree

Payoff	1/3(200) - 50	=	\$16.67
	1/3(50)	=	\$16.67

The hedged portfolio is riskless, thus it should get a return equal to the riskless rate.The Value of this portfolio at time 0=

$$= \frac{\$16.67}{1.10} = \$15.15$$

The value of the hedged portfolio is

$$(1/3)S_0 - C_0 = \$15.15$$

$$\text{and } S_0 = \$100 \quad \rightarrow \quad C_0 = \$18.18$$

17

Option pricing: binomial tree

This means that we can replicate the pay off of this call option investing in a portfolio that consists of 1/3 of the the stock and borrowing \$15.15.

pay off

$1/3S_0$-\$15.15

1/3 * 200 - \$15.15*1.1 = 66.67-16.17 =50

1/3 * 50 - \$15.15*1.1 = 16.67-16.17 = 0

This is the same as pay off from the call option!

So, $1/3S_0$-\$15.15 = C_0 =18.18

18

Option pricing (Algebra)

More generally

$$
S \xrightarrow{q} uS \qquad S \rightarrow dS
$$

$$
C \xrightarrow{q} C_u = \max[0, uS-X] \qquad C \rightarrow C_d = \max[0, dS-X]
$$

19

Option pricing (Algebra)

Stocks in the binomial model have equal probability of going up or down. Form a portfolio of Δ shares of stock and one call option (short).

$$
\Delta S - C \rightarrow \Delta\, uS - C_u \qquad \Delta S - C \rightarrow \Delta dS - C_d
$$

Pick Δ: so that the portfolio is riskless i.e.,

$$
\Delta uS - C_u = \Delta dS - C_d
$$

$$
\Delta (u-d)S = C_u - C_d \longrightarrow \Delta = \frac{C_u - C_d}{(u-d)S} \left(\approx \frac{dC}{dS}\right)
$$

20

Option pricing (Algebra)

Value of Portfolio at t = 0 : $\Delta S - C$

Value of Portfolio at t = 1 : $\Delta\, uS - C_u$ (for sure)

$$\rightarrow \quad \frac{\Delta uS - C_u}{1+r_f} = \Delta S - C$$

- Substituting Δ and defining $\quad p = \frac{r^* - d}{u-d}; \quad r^* = 1 + r_f$
- We have $\quad C = \frac{pC_u + (1-p)C_d}{1+r_f}$

21

Option Valuation: Binomial probability model

For

$$S_0 \begin{cases} S_u = uS \\ S_d = dS \end{cases} \qquad\qquad C \begin{cases} C_u \\ C_d \end{cases}$$

r_f = risk free rate for period t

$$C = \frac{pC_u + (1-p)C_d}{1+r_f}, \text{ where } \quad p = \frac{r^* - d}{u-d}; \quad r^* = 1 + r_f$$

If S_u and S_d are given then $\quad d = \frac{S_d}{S_o} \qquad u = \frac{S_u}{S_o}$

If the volatility is given then $d = e^{-\sigma\sqrt{t}}\ ; u = e^{\sigma\sqrt{t}}$

For put options, substitute C with P

22

Observations

- The option value does not depend on the rate of return of the underlying asset
- *p* can be interpreted as the probability that the option is going to end up in-the-money

23

Option Valuation: Binomial Method

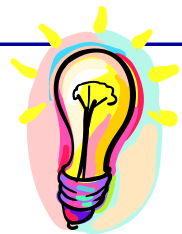
- Initial Example: S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%

$$p = \frac{1.1 - 0.5}{2.0 - 0.5} = \frac{0.6}{1.5} = 0.4$$

$$\longrightarrow C = \frac{0.4(\$50) + 0.6(\$0)}{1.10} = \$18.18$$

24

Concept check



- S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%. What is the price of a call option with X=100 ?

G) 18.18

Y) 30

R) 36.36

Concept check

- S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%. What is the price of a put option with X=100 ?

G) 18.18

Y) 30

R) 27.27

Concept check: solution

- S_0 = 100, S_1 = either 200 or 50, and riskfree rate =10%. What is the price of a put with X=100

$$p = \frac{1.1 - 0.5}{2.0 - 0.5} = \frac{0.6}{1.5} = 0.4$$

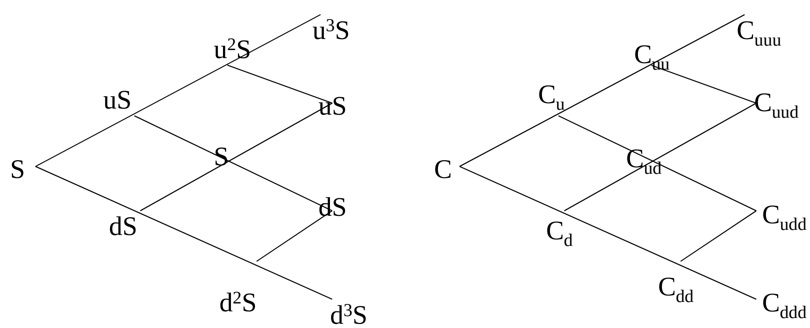
$$P_u = \max(0{,}100 - 200) = 0$$

$$P_d = \max(0{,}100 - 50) = 50$$

$$P = \frac{0.4(\$0) + 0.6(\$50)}{1.10} = \$27.27$$

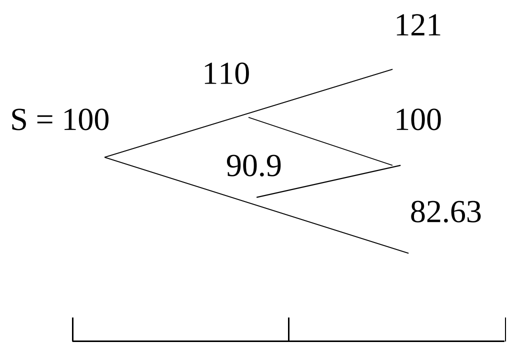
27

Binomial Method: Multi-step binomial model



28

Example: Call Option   X = $100



u = 1.10
d = 0.909
r* = 1.02

1 year

29

Sample: Call Option   X = $100

Price an at-the-money call option with X = $100

$$p = \frac{r^* - d}{u - d} = \frac{1.02 - 0.909}{1.10 - 0.909} = 0.5812$$

C_u: 21, 0

$$\rightarrow C_u = \frac{p(21) + (1-p)\,0}{1.02}$$

= $11.97

C_d: 0, 0

$$\rightarrow C_d = \frac{p(0) + (1-p)\,0}{1.02} \quad = \quad \$0$$

30

Sample: Call Option   X = $100

C_u = (11.97)

C_0

C_d = (0)

$$C_0 = \frac{pC_u + (1-p)C_d}{1.02} = \frac{0.5812(11.97) + (1-0.5812)(0)}{1.02} = 6.82$$

Option is worth $6.82

31

Summary

- Determinants of option price
- How to price options using a binomial model

32