Group members

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Question 0:

Consider the function f defined for all (x,y) such that

$$f\,(x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f\,(x, y; a)$.
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f\,(x^*, y^*; a)$, show the derivative of G with respect to a.
d. Calculate $\frac{\partial f\,(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1 - a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?
e. Determine the domain of (x,y) in the xy-plan where f is convex.

a) $\frac{\partial f}{\partial x} = 2\left(\frac{1}{2}x\right) - 1 + ay$

$= x - 1 + ay \quad \Rightarrow \quad x = 1 - ay$

$\frac{\partial f}{\partial y} = ax - a - 3\left(\frac{1}{3}y^2\right) + 2a^2y$

$= ax - a - y^2 + 2a^2y$

Put $x = 1-ay$ into $ax - a - y^2 + 2a^2y$

$a(1-ay) - a - y^2 + 2a^2y = 0$

$a - a^2y - a - y^2 + 2a^2y = 0$

$-y^2 + {}^2y = 0$

$-y(y - a^2) = 0$

$y^* = 0,\ a^2$

$x^* = 0,\ 1 - a^3$

$\therefore$ Stationary point of $f(x, y; a)$ is $(x^*, y^*) = (1-a^3, a^2)$

b) $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & -2y + 2a^2 \end{bmatrix}$

$|H_1| = 1 \quad > 0$

$|H_2| = (-2y + 2a^2) - (a^2)$

$= -2y + a^2 \quad < 0$

$|H_1| > 0$
$|H_2| < 0$

; $f(x,y;a)$ is positive definite
Local min
Global Min

c) $G(a) = \frac{1}{2}(1-a^3)^2 - (1-a^3) + a(a^2)\left[(1-a^3)-1\right] - \frac{1}{3}(a^2)^3 + a^2(a^2)^2$

$= \frac{1}{2}\left[1-2a^3+a^6\right] - 1 + a^3 - a^6 - \frac{1}{3}a^6 + a^6$

$= \frac{1}{2} - a^3 + \frac{1}{2}a^6 - 1 + a^3 - a^6 - \frac{1}{3}a^6 + a^6$

$= -\frac{1}{2} + \frac{1}{6}a^6$

$\frac{dG(a)}{da} = a^5$

d) $\frac{dF(x,y;a)}{da} = xy - y + 2ay^2$

$\left.\frac{dF(x,y;a)}{da}\right|_{\substack{x^*=1-a^3 \\ y^*=a^2}} = (1-a^3)(a^2) - a^2 + 2a(a^2)^2$

$= a^2 - a^5 - a^2 + 2a^5$

$= a^5$

e) $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & -2y+2a^2 \end{bmatrix}$

$|H_1| = 1 \quad > 0$

$|H_2| = (-2y+2a^2) - (a^2)$

$= -2y + a^2 \quad > 0$

To ensure that $|H_2|$ will be positive $y < \frac{a^2}{2}$

Question 2: Suppose that there are three groups of people who take Sky train to commute in Bangkok. The first group is students (s), the second group is senior citizens (old), and the third group is working-aged people. The demand for each group is given by the following equations:

Demand of students: $P_s = 8 - \left(\frac{1}{2}\right) Q_s$

Demand of senior citizens: $P_{old} = 16 - 2Q_{old}$

Demand of working-aged people: $P_w = 20 - Q_w$

The Sky train operator has a constant marginal cost at MC = \$4, and total cost at TC = 4Q + 10. Consider the following problems.

if they give you cost fn use it
if not make ur own

a) Determine the profit-maximizing level of output/price under third-degree price discrimination. Calculate the level of maximized profit.

b) Confirm your result in (a) with the second-order derivative test.

c) Calculate (i) consumer surplus for each of the three groups of consumers, and (ii) producer surplus.

d) Calculate the optimal level of total output if the Sky train operator can practice the first-degree price discrimination for each group of the consumers in the market.

A 3rd degree price discrimination

$$\pi^{dis} = P_{old} \cdot Q_o + P_s \cdot Q_s + P_w \cdot Q_w - (4(Q_o + Q_s + Q_w) + 10)$$

$$= (16 - 2Q_o)(Q_o) + (8 - \frac{Q_s}{2})(Q_s) + (20 - Q_w)(Q_w) - (4(Q_o + Q_s + Q_w) + 10)$$

$$= 16Q_o - 2Q_o^2 + 8Q_s - \frac{Q_s^2}{2} + 20Q_w - Q_w^2 - 4Q_o - 4Q_s - 4Q_w - 10$$

$$\pi^{dis}_s = \frac{\partial \pi^{dis}}{\partial s} = 8 - Q_s - 4 = 0 \qquad Q_s^* = 4 \qquad \Big| \qquad P_s = 8 - \frac{1}{2}(4) \qquad P_s^* = 6$$

$$\pi^{dis}_{old} = \frac{\partial \pi^{dis}}{\partial Q_o} = 16 - 4Q_o - 4 = 0 \qquad Q_o^* = \frac{12}{4} = 3 \qquad \Big| \qquad P_{old} = 16 - 2(3) \qquad P_o^* = 10$$

$$\pi^{dis}_w = \frac{\partial \pi^{dis}}{\partial w} = 20 - 2Q_w - 4 = 0 \qquad Q_w^* = \frac{16}{2} = 8 \qquad \Big| \qquad P_w = 20 - 8 \qquad P_w^* = 12$$

12

30 24 96

$$\pi^{dis}_{max} = (10 \cdot 3) + (6 \cdot 4) + (8 \cdot 12) - (4(4 + 3 + 8) + 10)$$

$$= 80$$

$\therefore$ \$80 is total profit under 3rd price discrimination

ⓑ S.O.C

$$H = \begin{bmatrix} \pi_{ss} & \pi_{so} & \pi_{sw} \\ \pi_{os} & \pi_{oo} & \pi_{ow} \\ \pi_{ws} & \pi_{wo} & \pi_{ww} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\pi^{dis}_{ss} = -1 \quad \pi^{dis}_{so} = 0 \quad \pi^{dis}_{sw} = 0$

$\pi^{dis}_{os} = 0 \quad \pi^{dis}_{oo} = -4 \quad \pi^{dis}_{ow} = 0$

$\pi^{dis}_{ws} = 0 \quad \pi^{dis}_{wo} = 0 \quad \pi^{dis}_{ww} = -2$

$|H_1| = -1 < 0$

$|H_2| = 4 > 0$

$B = 0 + 0 + 0$

$$|H_3| = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{vmatrix} -1 & 0 \\ 0 & -4 \\ 0 & 0 \end{vmatrix} \qquad A - B = -8$$

$A = -8 \quad 0 \quad 0$

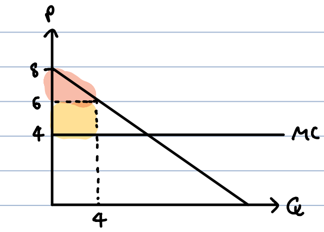
$|H_3| = -8 < 0$

$\therefore$ Hessian is negative definite $\forall\, Q_s, Q_o, Q_w$

$\partial^2 \pi < 0 \qquad \forall\, Q_s, Q_o, Q_w$

π^{dis} is globally concave

C



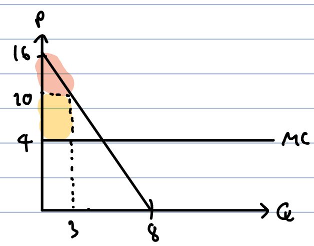
Student

Consumers Surplus $= \frac{1}{2} \times (8-6) \times 4$

$= 4$

Producer surplus $= (6-4)(4)$

$= 8$



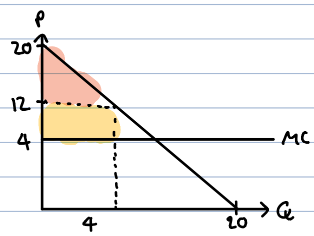
Senior

Consumer surplus $= \frac{1}{2} \times (16-10) \times 3$

$= 9$

Producer Surplus $= (10-4)(3)$

$= 18$



Working - age

Consumer Surplus $= \frac{1}{2} \times (20-12) \times 8$

$= 32$

Producer Surplus $= (12-4) \times 8$

$= 64$

D

1st degree price discrimination

$MR = MC$ → optimal condition

F.O.C Student

$8 - \frac{1}{2} Q_s = 4$

$-\frac{1}{2} Q_s = -4$

$Q_s = 8$

F.O.C Working age

$20 - Q_w = 4$

$Q_w = 16$

F.O.C Senior

$16 - 2Q_o = 4$

$Q_o = 6$

$Q^* = Q_o + Q_s + Q_w$

$Q^* = 30$ units

Question 4:

Suppose that the output *Q* of a firm depends on two inputs of the quantities *K* and *L*. The output level is determined by the production function

$$Q = 36K + 16L - 3K^2 - 2KL - L^2$$

a. Is the firm's production function strictly concave? Explain.

$$\frac{\partial Q}{\partial K} = 36 - 6K - 2L$$

$$\frac{\partial Q}{\partial L} = 16 - 2K - 2L$$

$$H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LL} & Q_{LK} \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ -2 & -2 \end{bmatrix}$$

$|H_1| = -6 < 0 \Rightarrow$ F is negative definite for all K and L

$|H_2| = (-6)(-2) - (-2)(-2) = 4 > 0$

$\therefore$ the production function is concave

b. Determine the optimal input (K*, L*) that maximizes the output level.

$\frac{\partial Q}{\partial K} = 36 - 6K - 2L$; $6K + 2L = 36$ —(1)

$\frac{\partial Q}{\partial L} = 16 - 2K - 2L$; $2K + 2L = 16$ —(2)

(1) − (2) ; $4K = 20$; $K^* = 5$ ※

plug-in $K=5$ into (1) ; $6(5) + 2L = 36$

$L^* = 3$ ※

$\therefore$ At $(K^*, L^*) = (5, 3)$ is the level of output maximization

c. Write down the firm's profit function when the price of Q is P and the per-unit factor prices of K and L are r and w, respectively, where both r and w are positive numbers. Find the levels of K^* and L^* that maximize the firm's profits.

$$\pi = TR - TC$$
$$\pi = PQ - wL - rk$$
$$\pi = P(36K + 16L - 3K^2 - 2kL - L^2) - wL - rk$$
$$\pi = 36KP + 16LP - 3K^2P - 2KLP - L^2P - wL - rk$$

$\frac{\partial \pi}{\partial K}$; $36P - 6KP - 2LP - r = 0$

$$36P - 6KP - r = 2LP$$
$$18 - 3K - \frac{r}{2P} = L \quad -(1)$$

$\frac{\partial \pi}{\partial L}$; $16P - 2KP - 2LP - w = 0$

$$16P - 2KP - w = 2LP$$
$$8 - K - \frac{w}{2P} = L \quad -(2)$$

(1) = (2) ; $18 - 3K - \frac{r}{2P} = 8 - K - \frac{w}{2P}$

$$10 - \frac{r}{2P} + \frac{w}{2P} = 2K$$
$$5 - \frac{r}{4P} + \frac{w}{4P} = K^* ※$$

Plug-in K into (1)

$$18 - 3\left(5 - \frac{r}{4P} + \frac{w}{4P}\right) - \frac{r}{2P} = L$$

$18 - 15 + \frac{3r}{4P} - \frac{3w}{4P} - \frac{r}{2P} = L$, $L^* = 3 + \frac{r}{4P} - \frac{3w}{4P}$ ※

d. Verify that the second-order sufficient conditions for maximum profits are satisfied.

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LL} & \pi_{LK} \end{bmatrix} = \begin{bmatrix} -6P & -2P \\ -2P & -2P \end{bmatrix}$$

$|H_1| = -6P < 0 \Rightarrow$ f is negative definite for all K and L

$|H_2| = (-6P)(-2P) - (-2P)(-2P) = 8P^2 > 0$

$\therefore$ K^* and L^* are the profit minimizer

e. Determine the effect of an increase in *r* on the firm's use of each input. (i.e. determine $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial r}$).

From 4c. we know that

$$K^* = 5 - \frac{r}{4P} + \frac{w}{4P}$$

$$L^* = 3 + \frac{r}{4P} - \frac{3w}{4P}$$

$\frac{\partial K^*}{\partial r} = -\frac{1}{4P}$; when r increases 1 unit, K will decrease by $\frac{1}{4P}$ unit

$\frac{\partial L^*}{\partial r} = \frac{1}{4P}$; when r increases 1 unit, L will increase by $\frac{1}{4P}$ unit