Forgotten Note

λ (lambda) refers to "shadow price" or "Lagrange Multiplier"

Forgotten Note

Example

From UMP,

$$L = U(x,y) + \lambda(\underbrace{P_x X + P_y Y - I}_{\text{constraint}})$$

Note that $\boxed{\dfrac{\partial L}{\partial \text{constraint}} = \lambda}$

Thus, λ tells what happens to the objective function when there is a small change in constraint.

UMP

1) Find Marshallian Demand

$$\max_{x,y} U(x,y) \quad \text{s.t.} \quad P_x x + P_y y = I$$

$$L = U(x,y) - \lambda(P_x X + P_y Y - I)$$

FOCs: $\frac{\partial L}{\partial x} = 0$; $MU_x - \lambda P_x = 0$

$$\lambda = \frac{MU_x}{P_x} \quad \text{—— (1)}$$

$\frac{\partial L}{\partial y} = 0$; $MU_y - \lambda P_y = 0$

$$\lambda = \frac{MU_y}{P_y} \quad \text{—— (2)}$$

$\frac{\partial L}{\partial \lambda} = 0$; $P_x X + P_y Y = I$ —— (3)

From (1) & (2) ; $\frac{MU_x}{P_x} = \frac{MU_y}{P_y}$ —— (4)

Suppose $U = xy$, so $MU_x = y$ and $MU_y = x$.

(4) becomes $\frac{y}{P_x} = \frac{x}{P_y}$ or $P_y Y = P_x X$ —— (5)

From (3) & (5) ; $P_x X + P_x X = I$

$$2P_x X = I$$

$$x^*(P_x, P_y, I) = \frac{I}{2P_x}$$

Similarly, $y^*(P_x, P_y, I) = \frac{I}{2P_y}$

} Marshallian Demand

Note : $\frac{\partial x}{\partial P_x} < 0 \rightarrow$ ordinary good (obeys law of demand)

$\frac{\partial X}{\partial I} > 0 \rightarrow$ normal good

2) Find Indirect Utility Function

$$V(P_x, P_y, I) = U(x^*, y^*)$$

We already have $\begin{cases} U = xy \\ x^*(P_x, P_y, I) = \dfrac{I}{2P_x} \\ y^*(P_x, P_y, I) = \dfrac{I}{2P_y} \end{cases}$

Thus, $V(P_x, P_y, I) = x^* \times y^*$

$$= \frac{I}{2P_x} \times \frac{I}{2P_y}$$

$$V(P_x, P_y, I) = \frac{1}{4} \frac{I^2}{P_x P_y}$$ } indirect U function

Note: $\dfrac{\partial V}{\partial P_x}, \dfrac{\partial V}{\partial P_y} < 0$

Higher $P_x, P_y \rightarrow$ Lower V

(lower maximum utility that consumer can achieve)

$\dfrac{\partial V}{\partial I} > 0$

Higher $I \rightarrow$ Higher V

(richer $\rightarrow$ higher maximum utility)

3) Verify Roy's Identity

Roy's Identity: $x^* = -\dfrac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial I}}$

Recall $V(P_x, P_y, I) = \frac{1}{4}\frac{I^2}{P_x P_y} = \frac{1}{4} I^2 P_x^{-1} P_y^{-1}$

$$\frac{\partial V}{\partial P_x} = -\frac{1}{4} I^2 P_x^{-2} P_y^{-1}$$

$$\frac{\partial V}{\partial I} = \frac{1}{2} I P_x^{-1} P_y^{-1}$$

$$-\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial I}} = -\frac{-\frac{1}{4} I^2 P_x^{-2} P_y^{-1}}{\frac{1}{2} I P_x^{-1} P_y^{-1}}$$

$$-\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial I}} = \cancel{-}\frac{\cancel{-}\frac{1}{4} I^{\cancel{2}} P_x^{-2} \cancel{P_y^{-1}}}{\frac{1}{2} \cancel{I} P_x^{-1} \cancel{P_y^{-1}}}$$

$$= \underbrace{\frac{I}{2P_x}}_{\text{Marshallian Demand}} = \vec{X}(P_x, P_y, I)$$

EMP

1) Find Hicksian Demand

$$\min_{x,y} \; P_x X + P_y Y \quad s.t. \quad U(x,y) = \bar{U}$$

$$L = P_x X + P_y Y - \lambda(U(x,y) - \bar{U})$$

FOCs: $\frac{\partial L}{\partial x} = 0$; $P_x - \lambda MU_x = 0$

$$\lambda = \frac{P_x}{MU_x} \quad \text{——} \quad (1)$$

$\frac{\partial L}{\partial y} = 0$; $P_y - \lambda MU_y = 0$

$$\lambda = \frac{P_y}{MU_y} \quad \text{——} \quad (2)$$

$\frac{\partial L}{\partial \lambda} = 0$; $U(x,y) = \bar{U}$ —— (3)

From (1) & (2); $\frac{MU_x}{P_x} = \frac{MU_y}{P_y}$ —— (4)

Suppose $U = xy$, so $MU_x = y$ and $MU_y = x$.

(3) becomes $xy = \bar{U}$ —— (5)

(4) becomes $\frac{y}{P_x} = \frac{x}{P_y}$ or $y = x\frac{P_x}{P_y}$ — (6)

From (5) & (6); $x^2 \frac{P_x}{P_y} = \bar{U}$

$$\hat{x}^*(P_x, P_y, \bar{U}) = \sqrt{\bar{U}\frac{P_y}{P_x}}$$

Similarly, $\hat{y}^*(P_x, P_y, \bar{U}) = \sqrt{\bar{U}\frac{P_x}{P_y}}$

Hicksian Demand

$$\hat{x}^*(P_x, P_y, \bar{U}) = \sqrt{\bar{U}\,\frac{P_y}{P_x}}$$

Note : $\frac{\partial \hat{x}}{\partial P_x} < 0 \longrightarrow$ Hicksian substitution effect

i.e. price changes, holding U constant

$\frac{\partial \hat{x}}{\partial P_y} > 0 \longrightarrow P_y \uparrow \longrightarrow x \uparrow$

$\frac{\partial \hat{x}}{\partial \bar{U}} > 0$

i.e. to increase U, more x must be consumed

2) Find Expenditure Function

$$E(P_x, P_y, \bar{U}) = P_x \hat{X}^* + P_y \hat{Y}^*$$

We already have

$$\hat{X}^*(P_x, P_y, \bar{U}) = \sqrt{\bar{U}\,\frac{P_y}{P_x}} = (\bar{U})^{\frac{1}{2}} P_y^{\frac{1}{2}} P_x^{-\frac{1}{2}}$$

$$\hat{Y}^*(P_x, P_y, \bar{U}) = \sqrt{\bar{U}\,\frac{P_x}{P_y}} = (\bar{U})^{\frac{1}{2}} P_x^{\frac{1}{2}} P_y^{-\frac{1}{2}}$$

Thus, $E = P_x (\bar{U})^{\frac{1}{2}} P_y^{\frac{1}{2}} P_x^{-\frac{1}{2}} + P_y (\bar{U})^{\frac{1}{2}} P_x^{\frac{1}{2}} P_y^{-\frac{1}{2}}$

$$= (\bar{U})^{\frac{1}{2}} P_y^{\frac{1}{2}} P_x^{\frac{1}{2}} + (\bar{U})^{\frac{1}{2}} P_x^{\frac{1}{2}} P_y^{\frac{1}{2}}$$

$$E(P_x, P_y, \bar{U}) = 2\sqrt{\bar{U} P_x P_y}$$ — expenditure function

Note: $\frac{\partial E}{\partial \bar{U}} > 0$

i.e. higher U requires more expenditure

$\frac{\partial E}{\partial P_x}, \frac{\partial E}{\partial P_y} > 0$

i.e. higher price requires more expenditure

3) Verify Shephard's Lemma

Shephard's Lemma: $\frac{\partial E}{\partial P_x} = \hat{X}^*$

Try this at home