Tutorial 3
Matrix Algebra

1. Find all solution to the following system of equations

   (a)

$$\begin{aligned} 3x+4y+z &= 1 \\ 2x+3y &= 0 \\ 4x+3y-z &= -2 \end{aligned}$$

(Ans: x=-3/7, y=2/7,z=8/7)

(b)

$$\begin{aligned} x+2y-z &= 2 \\ 2x+5y+2z &= -1 \\ 7x+17y+5z &= -1 \end{aligned}$$

Ans:

$$\begin{aligned} x &= 12+9z \\ y &= -5-4z \end{aligned}$$ (parametric form)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -5 \\ 0 \end{bmatrix} + C\begin{bmatrix} 9 \\ -4 \\ 1 \end{bmatrix}$$; C∈R (vectors form)

c)

$$\begin{aligned} x+10z &= 5 \\ 3x+y-4z &= -1 \\ 4x+y+6z &= 1 \end{aligned}$$

(ans: No solution)

2. Compute the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$

(ans: 2)

3. Find a condition on the number *a,b,c* such that the following system of equations is consistent. When that condition is satisfied, find all solutions (in term of *a,b, c*)

$$\begin{aligned} x+3y+z &= a \\ -x-2y+z &= b \\ 3x+7y\text{-}z &= c \end{aligned}$$

Ans: Consistent if c=a-2b

$$x = 5z - (2a + 3b)$$
$$y = (a + b) - 2z \quad \text{(Pararametric form)}$$
$$\text{Z} = \text{free variables}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -(2a+3b) \\ a+b \\ 0 \end{bmatrix} + C\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad C \in R \text{ (vectors form)}$$

4. Suppose a square matrix A satisfies $A = 2A^T$. Show that necessarily *A=0*.

Ans:
$$A = 2(2A^T)^T = 2(2(A^T)^T = 4A$$
$$3A = 0$$
$$A = 0$$

5. In each case find the matrix A

(a) $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

(b) $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 7 \\ \frac{-9}{2} & -5 \end{bmatrix}$

6. Find the inverse of the matrix

$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$

7. Find A if $\left(A^T - 2I\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

8. Suppose A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. Which of these matrix operations are allowed, and what are the shape of the results?

BA, A(B+C), ABD, AC+BD, ABABD

(Ans: **BA** (a 5 x5 matrix) **ABD** (a 3x1 matrix) and **ABABD** (a 3x1 matrix))

9. Compute AB where $A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ -4 & 7 \\ 2 & -5 \end{bmatrix}$. Show how you obtain the result using all three methods you study in the class.

Ans: $\begin{bmatrix} -6 & 21 \\ -14 & 14 \end{bmatrix}$

10. Find all solutions to the following system of linear equations:

$$\begin{aligned} x + y - 3z &= 3 \\ -2x - y &= -4 \\ 4x + 2y + 3z &= 7 \end{aligned}$$

Ans: x=2, y=0,z = -1/3

11. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system

a) $\left[\begin{array}{cc|c} 2 & 3 & h \\ 4 & 6 & 7 \end{array}\right]$ (Ans: h=7/2)

b) $\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 5 & h & -7 \end{array}\right]$ (Ans: $h \neq -15$)

12. For matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

determine the echelon form U, the basic variables, the free variables and the general solution to $\mathbf{A}x=0$. Then apply, elimination to $Ax=\mathbf{b}$, with b_1 and b_2 on the right side; find the conditions for $Ax=\mathbf{b}$ to be consistent and find the general solution. What is the rank of $\mathbf{A}$?

Ans: For $\mathbf{A}x=0$ ➔ $\mathbf{x} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + C_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ where $C_1, C_3, C_4 \in \mathrm{R}$

For $\mathbf{A}x=\mathbf{b}$ ➔

$$\mathbf{x}=\begin{bmatrix}0\\b_1\\0\\0\end{bmatrix}+C_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+C_3\begin{bmatrix}0\\-4\\1\\0\end{bmatrix}+C_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\text{ where } C_1, C_3, C_4 \in \mathrm{R}$$

13. a) Find all solutions to

$$\mathbf{Ux}=\begin{bmatrix}1&2&3&4\\0&0&1&2\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Ans: $\mathbf{x}=C_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+C_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}\text{ where } C_1, C_2 \in \mathrm{R}$

b) If the right side is changed from $\begin{bmatrix}0\\0\\0\end{bmatrix}$ to $\begin{bmatrix}a\\b\\0\end{bmatrix}$, what are the solutions?

Ans: $\mathbf{x}=\begin{bmatrix}a-3b\\1\\b\\0\end{bmatrix}+C_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+C_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}\text{ where } C_1, C_2 \in \mathrm{R}$

14. Use row operation to find

$$\begin{aligned}6x_1+x_2+x_3&=6\\5x_1+x_2+2x_3&=4\\4x_1+x_2-x_3&=-2\end{aligned}$$

Ans: $\begin{bmatrix}3\\-13\\1\end{bmatrix}$

15. Suppose

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}$$

but

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ has infinitely many solutions.}$$

a) Find all possible information about r, m and n. (r is the rank and A is of the size $m \times n$)

b) Find an example of such a matrix A with r, m and n all as small as possible.

Ans:

a) m=3, 0<r<m and r<n

b) r=1, m=3, n=2; $A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$

16. a) By elimination put A into its upper triangular form U. Which are the pivot columns and free columns?

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 8 & 5 & 2 \\ 1 & 5 & 3 & 1 \end{bmatrix}$$

(Ans: pivot columns 1,2 and free columns 3,4)

b) Describe specifically the vectors which are solutions to Ax=0

c) Does Ax=b have a solution for the right side $b = \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$? If it does, find one particular solution and then complete solution to this system Ax=b.

Ans:

$$x = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + c\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\\0\end{bmatrix} + d\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

17. a) Show that $A=\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}$ has inverse $C=\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$

Show that $A = \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}$ has inverse $C = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$.

We verify that $AC = I$ and $CA = I$:

$$AC = \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

and

$$CA = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

b) Show that $A=\begin{bmatrix}2 & -3\\0 & 0\end{bmatrix}$ has no inverse.

If $C = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ is *any* matrix, then $AC = \begin{bmatrix}2 & -3\\0 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}2a-3c & 2b-3d\\0 & 0\end{bmatrix}$. Since the (2, 2)-entry of AC is not 1, AC can never equal I for any choice of the matrix C.

18. a) Find the inverse of the matrix $A=\begin{bmatrix}1 & 4 & -1\\2 & 7 & 1\\1 & 3 & 0\end{bmatrix}$ using Gauss-Jordan method.

$$A^{-1} = \begin{bmatrix}-\frac{3}{2} & -\frac{3}{2} & \frac{11}{2}\\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2}\\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-3 & -3 & 11\\1 & 1 & -3\\-1 & 1 & -1\end{bmatrix}$$

b) Show that the matrix $A=\begin{bmatrix}1 & 3 & -2\\1 & 2 & 0\\2 & 8 & -8\end{bmatrix}$ is not invertible.

19. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to compute the inverse of the matrix A.

Ans:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(b) Use the answer in part (a) to solve the system $Ax = b$.

Ans:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

20. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 0 & 7 & 8 & 0 \\ 0 & 0 & 9 & 10 \end{bmatrix}.$$

(Ans: -530)

21. Suppose the 4 by 4 matrix M has four equal rows all containing a,b,c,d. Find determinant of this matrix

$$\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$

(Ans: 1+a+b+c+d)

Matrix: Additional questions set 1

1. John has inherited \$25,000 and put the money in three types of investment: a money market account, municipal bonds and a mutual fund. After one year, he received a total of \$1,620 in simple interest from the three investments. The money market paid 6% annually, the bonds paid 7% annually, and the mutually fund paid 8% annually. His investment in the bonds is \$6,000 more than that in the mutual funds. Find the amount John invested in each type of investment.

(a) Write down the matrix equation ($\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$) representing the problem.

(b) Use **Gauss-Jordan** method to obtain the inverse of the matrix $\underline{\mathbf{A}}$ in part (a)

(c) Use the answer from (b) to determine the amount of each type of investment.

ANS. $$\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.07 & 0.08 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 25000 \\ 1620 \\ 6000 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 15000 \\ 8000 \\ 2000 \end{bmatrix}$$

2. Let $\underline{\mathbf{A}}^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 5 \end{bmatrix}$ and $\underline{\mathbf{C}} = \begin{bmatrix} 1 & -1 & 0 \\ -4 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$

a) Find det $\underline{\mathbf{A}}^{-1}$ and adjoint of $\underline{\mathbf{A}}^{-1}$ (adj ($\underline{\mathbf{A}}^{-1}$))

b) Find $\underline{\mathbf{B}}^{-1}$ if $\left(\underline{\mathbf{A}}^{-1}(\underline{B}^{-1})^T - \underline{\mathbf{I}}\right)^T = \underline{\mathbf{C}}$.

c) Solve $\underline{\mathbf{A}}^T \underline{\mathbf{x}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

ANS. det $\underline{\mathbf{A}}^{-1} = 20$, adj $(\underline{\mathbf{A}}^{-1}) = \begin{bmatrix} 10 & -13 & -2 \\ 0 & 10 & 0 \\ 0 & -4 & 4 \end{bmatrix}$, $\underline{\mathbf{B}}^{-1} = \frac{1}{20}\begin{bmatrix} 33 & -10 & 4 \\ -85 & 30 & 0 \\ -11 & 10 & -8 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$

3. For a system of linear equation below where C and D are real constants

$$\begin{aligned} x_1 + 2x_2 + x_3 + 5x_5 + 2x_6 &= 1 + x_4 \\ 2x_1 + 5x_2 + 2x_3 + x_6 &= 2 - 3x_4 - 6x_5 \\ x_1 + 3x_2 + Cx_3 + 4x_4 + x_5 &= 5 - Dx_6 \end{aligned}$$

a) Write down the matrix equation ($\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$) and the augmented matrix that represents the above system of linear equations.

b) What are values of C and D that make this system inconsistent?

c) If $C = 1$, what are all possible values of D that make this system consistent?

d) If $D = 0$, choose a value of C that makes this system consistent. Also solve the given linear system with your specified value of C. **Write down the solution in vector form.**

e) If $C = 2$ and $D = 0$, is it possible for the linear system to have a unique solution? Give the reason to your answer.

ANS. b) C = 1 and D = -1
c) D $\neq$ -1
d) solution depends on the value of C chosen

4. Given $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 1 & 2 & 0 & 5 \end{bmatrix}$,

a) Use **cofactor expansion method together with row operations** to determine $|\mathbf{A}|$ and $|\mathbf{A}^{-1}|$

b) Let $\mathbf{B} = \begin{bmatrix} 0 & a & 2 \\ 0 & 0 & 9 \\ a & 1 & 2 \end{bmatrix}$; $a > 0$, if det **B** = 81, find the value of a

c) Determine det **C** if $(\det \mathbf{B})^{\frac{1}{2}} \det(\mathbf{A}^2 \mathbf{A}^T) = \dfrac{\det(\mathbf{C}^3)}{\det(\mathbf{A}^{-1})}$

d) Using **Cramer's rule** to solve for $\underline{\mathbf{B}}\underline{\mathbf{x}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

ANS. $|\mathbf{A}| = -9$, $|\mathbf{A}^{-1}| = 1/9$, $a = 3$, det $\mathbf{C} = 9$, $\underline{\mathrm{x}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{0} \end{bmatrix}$

5. If A is 4×4 matrix such that $\underline{\mathbf{A}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and $\underline{\mathbf{A}} \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 5/2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1/2 \end{bmatrix}$,

what is the product $\underline{\mathbf{A}} \begin{bmatrix} 1 \\ 11 \\ 5 \\ 11 \end{bmatrix}$? (Hints: you can obtain $\underline{\mathbf{A}} \begin{bmatrix} 1 \\ 11 \\ 5 \\ 11 \end{bmatrix}$ without finding the matrix $\underline{\mathbf{A}}$)

ANS. $\begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}$

6. State **True or False** and give **reason** to your answer briefly.

- Every matrix is row equivalent to a unique matrix in reduced echelon form.
- If a system of linear equations has no free variables, then it has a unique solution.
- If $\underline{\mathbf{A}}$ is an 4x6 matrix and the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$ is consistent for every $\underline{\mathbf{b}}$, then $\underline{\mathbf{A}}$ has 6 pivot columns.
- If $\underline{\mathbf{A}}$ is an *m*x*n* matrix, if the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$ has at least two different solutions, and if the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{c}}$ is consistent, then the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{c}}$ has many solutions.
- An economist solves a homogeneous system of 50 equations with 54 variables and finds that exactly 4 of the unknowns are free variables. The economist can be certain that for any associated nonhomogeneous system (with the same coefficients) always have solutions?

Matrix: Additional questions set 2

1. A zoo veterinarian can purchase animal food of three different types: A, B and C. Each food comes in the same size bag and the number of grams of each of three nutrients (N1, N2, N3) in each bag is summarised in the following table

	A	B	C
N1	5	5	10
N2	10	5	30
N3	15	15	10

For one animal, the veterinarian determines that she needs to combine the food types to get 10,000g of N1, 20,000 of N2 and 20,000g of N3

(a) Write down the matrix equation ($\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$) representing the problem.

(b) Use **<u>Gauss-Jordan</u>** method to obtain the inverse of the matrix $\underline{\mathbf{A}}$ in part (a)

(c) Use the answer from (b) to determine how many bags of each type of food should she order?

(d) If she would like to change the number of grams for each nutrients to be 8000g of N1, 18,400 of N2 and 19,000 of N3, determine how many bags of each type of food that she need for her new order.

Ans., b) $A^{-1} = \begin{bmatrix} -0.8 & 0.2 & 0.2 \\ 0.7 & -0.2 & -0.1 \\ 0.15 & 0 & -0.05 \end{bmatrix}$, c) $\begin{bmatrix} 0 \\ 1000 \\ 500 \end{bmatrix}$, d) $\begin{bmatrix} 1080 \\ 20 \\ 250 \end{bmatrix}$

2. Given $\underline{\mathbf{A}} = \begin{bmatrix} a^3 & a^2 & a & 1 \\ a & 1 & a & a^2 \\ a^2 & a & 1 & a \\ 1 & a & a^2 & a^3 \end{bmatrix}$; $\underline{\mathbf{B}} = \begin{bmatrix} 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \end{bmatrix}$ and $\underline{\mathbf{C}} = \begin{bmatrix} 2 & 4 & -8 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}$

(a) If $\det(-18\underline{\mathbf{A}}) = \det(\underline{\mathbf{B}}^{-1}\underline{\mathbf{D}}^3)\det\underline{\mathbf{C}}$, find the determinant of $\underline{\mathbf{D}}$.

(b) Find the adjoint of $\underline{\mathbf{C}}$ (*adj* $\underline{\mathbf{C}}$) and use it to show that $\underline{\mathbf{C}}^{-1} = \begin{bmatrix} \frac{1}{2} & -2 & 2 \\ 1 & 1 & 1 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$.

(c) Find the complete solution to $\underline{\mathbf{C}}^T \underline{\mathbf{x}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Ans. a) -108(1-a^2), b) $\begin{bmatrix} 1 & -4 & 4 \\ 2 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix}$, c) $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

3. Given a matrix equation

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}} \text{ ; where } \underline{\mathbf{A}} \text{ is a 4 by 3 matrix}$$

If $\underline{\mathbf{x}} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$ then $\underline{\mathbf{b}} = \begin{bmatrix} -4 \\ -3 \\ -7 \\ -1 \end{bmatrix}$. However, if $\underline{\mathbf{x}} = c\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, where $C \in \Re$, then $\underline{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(a) What is the 3rd column if $\underline{\mathbf{A}}$?

(b) What is the 2nd column of $\underline{\mathbf{A}}$?

Ans. a) $\begin{bmatrix} -2 \\ -3/2 \\ -7/2 \\ -1/2 \end{bmatrix}$, b) $\begin{bmatrix} -1 \\ -3/4 \\ -7/4 \\ -1/4 \end{bmatrix}$

4. (a) Given

$$\underline{\mathbf{A}}\begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = \begin{bmatrix} -3y-6z+4w+9v \\ -x-2y-z+3w+v \\ -2x-3y+3w+Dv \\ x+4y+5z-9w-7v \end{bmatrix}, \text{ where } D \text{ is a constant.}$$

And $\underline{\mathbf{C}}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$., where $\underline{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

Determine matrix $\underline{\mathbf{A}}$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. Show that $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is a summation (combination) of multiple of column of $\underline{\mathbf{C}}$.

(b) Obtain echelon form of matrix $\underline{\mathbf{A}}$. What is the maximum number of pivots that this matrix can have? What is the value of D that will make this matrix a rank 4 matrix?

(c) If $\underline{\mathbf{A}}\begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, can this linear system have unique solution? Explain why?

(d) If $\underline{\mathbf{A}}\begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, what value of D that will make the matrix $\underline{\mathbf{A}}$ a rank 3 matrix? Find the solution and give the solution in a vector form.

(e) If $\underline{\mathbf{A}}\begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \end{bmatrix}$ where $\underline{\mathbf{A}}$ is a rank 3 matrix, is this linear system consistent? If not, explain why? If so, give the solution in a vector form?

Ans. a) $\underline{\mathbf{A}} = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & D \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix}$, b) 4 , D≠-1 , c) no because ,

d) D = -1, $\underline{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_5 \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *where C_3 and C_5 are real numbers*

e) The system is inconsistent

5. Suppose $\underline{\mathbf{A}}$ is an m by n matrix of rank r

(a) Under what conditions on m, n and r does $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$ have infinitely many solution?

(b) If the only solution to $\underline{\mathbf{A}}\underline{\mathbf{x}} = 0$ is $\underline{\mathbf{x}} = 0$, what is the rank of matrix $\underline{\mathbf{A}}$ in relation to m and/or n? Explain why?

(c) If $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$ always has at least one solution, show that the only solution to $\underline{\mathbf{A}}^T\underline{\mathbf{y}} = 0$ is $\underline{\mathbf{y}} = 0$. (*hint: what is the rank?)*

6. State whether each of the following statement is **True or False**. Explain your answer briefly. (*Answer without explanation will not have any score*)

(a) If rank of $\underline{\mathbf{B}}_{3x2}$ is 2 then there is no solution to $\underline{\mathbf{B}^T\mathbf{x}} = [1\ 1]^T$.

(b) Any system of n linear equations in n variables has at most n solutions.

(c) If a system of linear equations has no free variables, then it has a unique solution.

(d) If $\underline{\mathbf{A}}$ is an m by n matrix and the equation $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$ is consistent for every $\underline{\mathbf{b}}$, then $\underline{\mathbf{A}}$ has m pivot columns.

(e) If $\underline{\mathbf{C}}$ is 3 by 4 matrix and det $\underline{\mathbf{C}}$ =0 then echelon form of $\underline{\mathbf{C}}$ will have a zero row.

(f) If $\underline{\mathbf{D}}$ is 5 by 5 matrix, it is invertible provided that det $\underline{\mathbf{D}}$ is a non-zero quantity.

Matrix: Additional questions set 3

1. For a given vector equation $x_1\begin{bmatrix}1\\4\\-2\end{bmatrix}+x_2\begin{bmatrix}2\\-9\\-2\end{bmatrix}+x_3\begin{bmatrix}3\\-5\\-4\end{bmatrix}+x_4\begin{bmatrix}2\\8\\-4\end{bmatrix}+x_5\begin{bmatrix}4\\-1\\C\end{bmatrix}=\begin{bmatrix}4\\-18\\D\end{bmatrix}$

   a) Write down the matrix equation ($\underline{\mathbf{A}}\underline{\mathbf{x}}=\underline{\mathbf{b}}$).
   b) What is the value of C that make matrix $\underline{\mathbf{A}}$ a rank **2** matrix?
   c) With the value of C in b), what is the value of D which makes this linear system consistent? **Find the solution and give the solution in a vector form.**
   d) What is the value of C that make matrix $\underline{\mathbf{A}}$ a rank **3** matrix? Can this rank 3 matrix $\underline{\mathbf{A}}$ have no solution? Explain why?

Ans. b) C = -6, c). D = -4, $\underline{x}=\begin{bmatrix}0\\2\\0\\0\\0\end{bmatrix}+C_3\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}+C_4\begin{bmatrix}-2\\0\\0\\1\\0\end{bmatrix}+C_5\begin{bmatrix}-2\\-1\\0\\0\\1\end{bmatrix}$, d) $C \neq -6$

2. For a given system of linear equations:

$$x_1 = 3$$
$$4x_1 + x_2 = 1$$
$$2x_1 - x_2 = K$$

   a) What is the value of K that will make this linear system have a unique solution? **Find the solution.**
   b) What is the value of K that will make this linear system have no solution?
   c) Is it possible that this linear system has infinitely many solutions? Explain why?

Ans. a). $K = 17, \underline{x} = \begin{bmatrix}3\\-11\end{bmatrix}$, b). $K \neq 17$

3. A health spa customizes the diet and vitamin supplements of each of its clients. The spa offers three different vitamin supplements (X,Y and Z) each containing different percentages of the recommended daily allowance (RDA) of vitamins A, C and D. The percentages of RDA of vitamins A, C and D provided by each tablet of supplements X, Y and Z are summarized in the table below

		Supplement		
		X	Y	Z
Vitamins	A	10%	20%	30%
	C	10%	30%	50%
	D	10%	50%	60%

The spa staff determines that one client should take 150% of RDA of vitamin A, 180% of the RDA of vitamin C and 210% of the RDA of vitamin D each day.

a) Write down the matrix equation ($\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$) representing the problem. Suppose this client should take x tablets of supplement X , y tablets of supplement Y and z tablets of supplement Z

b) Use **<u>Gauss-Jordan</u>** method to obtain the inverse of the matrix $\underline{\mathbf{A}}$ in part a).

c) Determine how many tablets of each supplement a client should take each day in order to obtain the recommended percentages of each vitamin. (hints: solve for *x, y* and *z*)

Ans. $A^{-1} = \begin{bmatrix} 7/30 & -0.1 & -1/30 \\ 1/30 & -0.1 & 2/30 \\ -2/30 & 0.1 & -1/30 \end{bmatrix}$, x = 10, y = 1, z = 1

4. Given

$$\underline{\mathbf{A}} = \begin{bmatrix} x^2 & y^2 & z^2 & 1 \\ 3 & 9 & 8 & 1 \\ 3 & 4 & 5 & 1 \\ 3 & 7 & 5 & 1 \end{bmatrix}, \underline{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 1 & -13 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 3 \end{bmatrix} \text{ and } \underline{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & 0 & -4 & 0 \\ -17 & \frac{1}{8} & -7 & -\frac{1}{4} \end{bmatrix}$$

a) If $y = 4$ and $z = 13$, determine all possible values of x such that $\det \underline{\mathbf{A}} = \det(\underline{\mathbf{B}}\underline{\mathbf{C}}^2)$.

b) If $x = 3$, $y = 1$ and $z = 2$, determine $\det \underline{\mathbf{A}}$, $\det(-2\underline{\mathbf{A}}^T)$ and $\det \underline{\mathbf{D}}$, provided that $\underline{\mathbf{B}} = 9\underline{\mathbf{A}}\underline{\mathbf{D}}^3\underline{\mathbf{C}}^{-1}$.

c) If $x = 3$, $y = 1$ and $z = 2$, use Cramer's rule to solve $\underline{\mathbf{A}}\underline{\mathbf{x}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Ans. a). x = ±2 , b). $|D| = 1/3$, c) $\underline{x} = \begin{bmatrix} 1/9 \\ 0 \\ 2/3 \\ -11/3 \end{bmatrix}$

5. Suppose that you were asked to write an exam for this class and you wish to produce a linear system of three equations with infinitely many solutions. If the first two equations are $2x + 3y - z = 5$ and $x + 2y + z = 3$, what might the third equation be? What if you wanted a system with no solution?