Group 8

1. a)

min cost : $wL + rK$

s.t. production function $\bar{Q} = f(K,L) = \alpha\sqrt{K} + \beta L$

$\mathcal{L} = wL + rK + \lambda(\bar{Q} - \alpha\sqrt{K} - \beta L)$

F.O.C.S

$[K] \rightarrow \mathcal{L}_K = r - \lambda \frac{1}{2}\alpha K^{-\frac{1}{2}} = 0 \rightarrow ①$

$[L] \rightarrow \mathcal{L}_L = w - \lambda\beta = 0 \rightarrow ②$

$[\lambda] \rightarrow \mathcal{L}_\lambda = \bar{Q} - \alpha\sqrt{K} - \beta L \rightarrow ③$

$\frac{①}{②}$; $\frac{r}{w} = \frac{\cancel{\lambda}\frac{1}{2}\alpha K^{-\frac{1}{2}}}{\cancel{\lambda}\beta} = \frac{1}{2}\frac{\alpha}{\beta\sqrt{K}} \rightarrow ④$

$④^2$; $\left(\frac{r}{w}\right)^2 = \frac{\alpha^2}{4\beta^2 K}$

$= \frac{\alpha^2}{4\beta^2\left(\frac{r}{w}\right)^2}$

$K^* = \left(\frac{w\alpha}{2\beta r}\right)^2$

sub into

③ ; $\bar{Q} - \alpha\sqrt{\left(\frac{w\alpha}{2\beta r}\right)^2} - \beta L = 0$

$\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right) - \beta L = 0$

$\frac{\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right)}{\beta} = L^*$

$K^* = \left(\frac{w\alpha}{2\beta r}\right)^2 \quad L^* = \frac{\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right)}{\beta}$

1b.)

$$H = \begin{bmatrix} L_{hh} & L_{hk} & L_{hL} \\ L_{kh} & L_{kk} & L_{kL} \\ L_{Lh} & L_{Lk} & L_{LL} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \frac{1}{2} k^{-\frac{1}{2}} & -\beta \\ -\alpha \frac{1}{2} k^{-\frac{1}{2}} & \frac{1}{4} h \alpha k^{-\frac{3}{2}} & 0 \\ -\beta & 0 & 0 \end{bmatrix}$$

$$\left(-\beta \cdot -\beta \cdot \tfrac{1}{4} h \alpha k^{-\frac{3}{2}}\right) \qquad 0 \qquad 0$$

$$= \left[\begin{array}{ccc|cc} 0 & -\alpha \frac{1}{2} k^{-\frac{1}{2}} & -\beta & 0 & -\alpha \frac{1}{2} k^{-\frac{1}{2}} \\ -\alpha \frac{1}{2} k^{-\frac{1}{2}} & \frac{1}{4} h \alpha k^{-\frac{3}{2}} & 0 & -\alpha \frac{1}{2} k^{-\frac{1}{2}} & \frac{1}{4} h \alpha k^{-\frac{3}{2}} \\ -\beta & 0 & 0 & -\beta & 0 \end{array}\right]$$

$$0 \qquad 0 \qquad 0$$

$$= (0 + 0 + 0) - \left[\beta^2 \cdot \left(\tfrac{1}{4} h \alpha k^{-\frac{3}{2}}\right)\right]$$

$$= -\left[\beta^2 \cdot \left(\tfrac{1}{4} h \alpha k^{-\frac{3}{2}}\right)\right] < 0$$

Convex, confirming the result.

1c.)

$$k^* = \left(\frac{w\alpha}{2\beta r}\right)^2 \qquad L^* = \frac{\bar{Q} - \frac{w\alpha^2}{2\beta r}}{\beta}$$

$$\left(\frac{w\alpha}{2\beta r}\right)^2 > 0 \qquad \bar{Q} - \frac{w\alpha^2}{2\beta r} > 0$$

$$\beta, r \neq 0 \qquad \beta, r \neq 0$$

1d.) $k^* = \left(\frac{w\alpha}{2\beta r}\right)^2$ $L^* = \frac{\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right)}{\beta}$

$$C^* = w\left[\frac{\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right)}{\beta}\right] + r\left(\frac{w\alpha}{2\beta r}\right)^2$$

Prove

$$\lambda^* = \frac{\partial C^*}{\partial \bar{Q}}$$

$$[L] \rightarrow \mathcal{L}_L = w - \lambda\beta = 0 \rightarrow \lambda^* = \frac{w}{\beta}$$

$$mc = \frac{\partial C^*}{\partial \bar{Q}} = \frac{w}{\beta}$$

2a.)

$$du(x,y) = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial x}dy$$

$$= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot x \cdot dx + \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y \cdot dy$$

$$= (x^2+y^2)^{-\frac{1}{2}}dx + (x^2+y^2)^{-\frac{1}{2}}dy$$

2b.)

Marshallian demand function: $\max U(x,y) = (x^2+y^2)^{\frac{1}{2}}$

s.t. $P_x \cdot x + P_y \cdot y = M$

$$\mathcal{L} = (x^2+y^2)^{\frac{1}{2}} + \lambda(m - P_x \cdot x - P_y \cdot y)$$

FOCS

$$\frac{\partial \mathcal{L}}{\partial x} = 0; \quad \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x + \lambda(-P_x) = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0; \quad \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y + \lambda(-P_y) = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0; \quad m - P_x \cdot x - P_y \cdot y = 0 \quad (3)$$

$$(1) \rightarrow (x^2+y^2)^{-\frac{1}{2}} \cdot x = \lambda P_x \rightarrow (4)$$

$$(2) \rightarrow (x^2+y^2)^{-\frac{1}{2}} \cdot y = \lambda P_y \rightarrow (5)$$

$$\frac{(4)}{(5)} = \frac{\cancel{(x^2+y^2)^{-\frac{1}{2}}}}{\cancel{(x^2+y^2)^{-\frac{1}{2}}}} \cdot \frac{x}{y} = \frac{\lambda P_x}{\lambda P_y}$$

$$y = x \frac{P_y}{P_x} \quad (6)$$

2b continued) sub ⑥ → ③

$$m - p_x \cdot x - p_y \cdot x \cdot \frac{p_y}{p_x} = 0$$

$$M = p_x \cdot x + p_y^2 \cdot \frac{x}{p_x}$$

$$M = x\left(p_x + \frac{p_y^2}{p_x}\right)$$

$$\frac{M}{\left(p_x + \frac{p_y^2}{p_x}\right)} = x$$

$$x^* = \frac{M \cdot p_x}{p_x^2 + p_y^2}$$

$$M - p_x \cdot \left(\frac{p_x}{p_y} \cdot y\right) - p_y \cdot y = 0$$

$$M = \frac{p_x^2}{p_y} \cdot y + p_y \cdot y$$

$$M = y\left(\frac{p_x^2}{p_y} + p_y\right)$$

$$\frac{M \cdot p_y}{p_x^2 + p_y^2} = y^*$$

Marshallian Demand Function $(x^*, y^*) = \frac{m \cdot p_x}{p_x^2 + p_y^2}, \frac{M \cdot p_y}{p_x^2 + p_y^2}$

2C.) $$\frac{dx^*}{dP_x} = \frac{(P_x^2 + P_y^2)(M) - (M \cdot P_x)(2P_x)}{(P_x^2 + P_y^2)^2}$$

$$= \frac{MP_x^2 + MP_y^2 - 2MP_x^2}{(P_x^2 + P_y^2)^2}$$

$$= \frac{MP_y^2 - MP_x^2}{(P_x^2 + P_y^2)^2}$$

$$= -\frac{M(P_x^2 - P_y^2)}{(P_x^2 + P_y^2)^2}$$

1. x^* and y^* are positive

2. demand will decrease as price increase for normal goods.

3. $$\text{slope} = -\frac{\Delta y}{\Delta x} = \frac{M \cdot P_y}{\cancel{P_x^2 + P_y^2}} \cdot \frac{\cancel{P_x^2 + P_y^2}}{M \cdot P_x} = -\frac{P_y}{P_x}$$

4. price of x should be more than price of y for equation to satisfy

2d) $y^* = \frac{M \cdot P_y}{P_x^2 + P_y^2}$ $\quad \frac{dy^*}{dP_x} = \frac{(P_x^2+P_y^2)(0) - (M \cdot P_y)(2P_x)}{(P_x^2+P_y^2)^2}$

$$= -\frac{2MP_xP_y}{(P_x^2+P_y^2)^2}$$

When price of good x $\uparrow$ by 1 unit

demand for good y $\downarrow$ by $\frac{2MP_xP_y}{(P_x^2+P_y^2)^2}$

Thus, good x and good y are complement products

2e) $x^* = \frac{M \cdot P_x}{P_x^2+P_y^2} = \frac{300 \cdot 1}{1^2+1^2} = 150$

$$y^* = \frac{M \cdot P_y}{P_x^2+P_y^2} = \frac{300 \cdot 1}{1^2+1^2} = 150$$

$$\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x + \lambda(-P_x) = 0$$

$$\frac{x}{\sqrt{x^2+y^2}} = \lambda(1)$$

$$\frac{150}{\sqrt{150^2+150^2}} = \lambda$$

$$\lambda^* = 0.7071$$

2f) $\lambda^* = \frac{\Delta u}{\Delta M}$; $\Delta M = 310 - 300 = 10$

$\lambda^* \cdot \Delta M = \Delta u$

$0.7071(10) = \Delta u$

$7.071 = \Delta u$

old $u = (x^2+y^2)^{\frac{1}{2}} = (150^2+150^2)^{\frac{1}{2}} = 212.132$

new u = old $u + \Delta u = 212.132 + 7.071$

$= 219.2030$