# Practice Problems for Midterm Exam

1. Determine whether the statement forms are logically equivalent. In each case, construct a truth table to justify your answer.
   (a) $(p \wedge q) \to r, \qquad (p \to r) \wedge (q \to r)$
   (b) $p \to (q \to r), \qquad (p \to q) \to r$
   (c) $p \to q \vee r, \qquad p \wedge \sim q \to r, \qquad p \wedge \sim r \to q.$
2. Determine whether or not the statement $\sim q \wedge p \to \sim q$ is a tautology.
3. Let $p$ and $q$ be statements such that $p \leftrightarrow q$ is true. Find the truth values of each of the followings statement forms and provide some justifications.
   (a) $p \to q$ (b) $\sim p \to \sim q$ (c) $\sim p \wedge q$ (d) $p \vee \sim q$ (e) $\sim p \leftrightarrow q$
4. Consider the following statement.

   *If its color is green and it is edible, then it is a vegetable or it is a fruit.*

   (a) Write the **negation** of the above statement.
   (b) Write the **contrapositive**, **inverse**, and **converse** of the above statement.
5. Use truth tables to determine whether the argument forms are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

   (a)
   $p \to q$
   $q \to p$
   $\therefore\ p \vee q$

   (b)
   $p$
   $p \to q$
   $\sim q \vee r$
   $\therefore\ r$

   (c)
   If it rains, then I stay home.
   If it does not rain, then I go shopping.
   ∴ I stay home or I go shopping.

   (d)
   If it rains, then I stay home.
   If it does not rain, then I go shopping.
   I do not go shopping.
   ∴ It rains.

6. Determine the truth value of each of these statements. Explain your answer.
   (a) $\forall n \in \mathbb{Z},\ \ n - 1 < n$ (b) $\forall n \in \mathbb{Z},\ \ n \leq 10n$
   (c) $\exists n \in \mathbb{Z},\ \ 2n = -n$ (d) $\exists n \in \mathbb{Z}^-,\ \ n = \frac{1}{n}$
7. Let $\mathbb{R}$ be the domain of $x$. Determine the **truth set** for each of these statements.
   (a) $P(x)$ : "$x + 1 < 2x$" (b) $P(x)$ : "$x^2 < 4$ and $x \leq 0$"

8. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, determine the truth values of the following statements. Explain you answer.

   (a) $Q(1, 1)$ (b) $Q(2, 0)$ (c) $\forall y, Q(1, y)$

   (d) $\exists x, Q(x, 2)$ (e) $\forall x \exists y, Q(x, y)$ (f) $\forall y \exists x, Q(x, y)$

9. Let $Q(x, y, z)$ be the statement "$x + y = z$." Let the domain of all variables be the set of all real numbers. Determine the truth value of the statement $\exists z \forall x \forall y,\ Q(x, y, z)$. Explain your answer.

10. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "$x$ is a hummingbird," "$x$ is large," "$x$ lives on honey," and "$x$ is colorful," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ and determine whether this argument is valid or invalid by applying the equivalences of statement forms and the inference rule(s).

    "All hummingbirds are colorful."
    "No large birds live on honey."
    "Birds that do not live on honey are not colorful."
    $\therefore$ "Hummingbirds are not large birds."

11. Let $\mathbb{Z}^+$ be the domain of $x$. Determine whether the following statements are true or false. Give a counterexample for each false statement.
    (a) $\sqrt{x} > 1 \Rightarrow x^2 > 23$, (b) $\sqrt{x} > 2 \Leftrightarrow x^2 > 23$.

12. Write a negation for each statement without using *the negation symbol* "$\sim$."

    (a) $\exists x \in \mathbb{R}$, $(x - 2)(x + 1) > 0$ if and only if $x > 2$ or $x < -1$.
    (b) $\forall \varepsilon \exists \delta \forall x\, (|x - a| < \delta \rightarrow |f(x) - L| < \varepsilon)$

13. Show that each of the following arguments is valid by **universal modus ponens** , **universal modus tollens** or **universal transitivity**, or show that it is invalid from the **converse error** or the **inverse error**. In addition, use also the **diagram** to confirm that each argument is valid or invalid.

    (a) All rabbits like vegetable. My pet is not a rabbit. Therefore, my pet does not like vegetable.
    (b) Everyone who eats fruit every day is healthy. Linda is not healthy. Therefore, Linda does not eat fruit every day.

14. (a) Prove the statement:
    "There is a pair of real numbers $x$ and $y$ such that $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$ . "

    (b) Disprove the statement: "For all real numbers $x$ and $y$, $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$ . "

15. Show that "for any integer $n$, if $n^3 + 5$ is odd, then $n$ is even," by using
    a) a proof by contraposition,
    b) a proof by contradiction.

16. Prove by the **method of exhaustion** that " $n^2 + 1 \geq 2^n$ for any positive integer $n$ with $1 \leq n \leq 4$."

17. Use the **proof by cases** to show that " for any integer $n$, $n^2 \geq n$."
    [Hint: Consider 3 cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$ ]

18. Consider the statement: for $n \geq 1$,

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

    Suppose we want to prove the above statement by **mathematical induction**.

    (a) What is $P(n)$?
    (b) Write $P(1)$: Is $P(1)$ true?
    (c) Write $P(k)$:
    (d) Write $P(k+1)$:
    (e) Prove the above statement: $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$, by using **mathematical induction**.

19. Use mathematical induction proof to show that

$$n! < n^n,$$

    for any integer $n$ that is greater than 1.