Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score......................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

Let $S_t = W_t - C_t$, $R_t = R_f + w_t^*(R_{it} - R_f)$

$U_c = E_{T-1}[B_W R_{T-1}]$

$\delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}[\delta^T W_T^{-\gamma} R_{T-1}] = E_{T-1}[\delta^T (S_{T-1} R_{T-1})^{-\gamma} R_{T-1}]$

$= \delta^T E_{T-1}[R_{T-1}^{1-\gamma}](W_{T-1} - C_{T-1})^{-\gamma}$

So $C_{T-1} = \dfrac{\delta E_{T-1}[R_{T-1}^{1-\gamma}]^{-\frac{1}{\gamma}}}{1+(\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{-\frac{1}{\gamma}}} W_{T-1}$

$= \dfrac{a_1}{1+a_1} \dfrac{W_{T-1}}{1-\gamma}$

Let $a_1 = (\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{-1/\gamma}$

FOC respect to the optimal weight

$E_{T-1}[(S_{T-1} R_{T-1})^{-\gamma} R_{r,T-1}] = \delta^T R_{f,T-1} E_{T-1}[(S_{T-1} R_{T-1})^{-\gamma}]$

$E[R_{T-1}^{-\gamma} R_{r,T-1}] = R_{f,T-1} E[R_{T-1}^{\gamma-1}]$

W_{T-1}^* doesn't depend on wealth or consumption its depend a return risky asset.

Let $S_t = W_t - C_t$, $R_t = R_f + w_t^*(R_{it} - R_f)$

$$U_c = E_{T-1}[B_W R_{T-1}]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}[\delta^T W_T^{-\gamma} R_{T-1}] = E_{T-1}[\delta^T (S_{T-1} R_{T-1})^{-\gamma} R_{T-1}]$$

$$= \delta^T E_{T-1}[R_{T-1}^{1-\gamma}](W_{T-1} - C_{T-1})^{-\gamma}$$

So
$$C_{T-1} = \frac{\delta E_{T-1}[R_{T-1}^{1-\gamma}](W_{T-1} - C_{T-1})^{-\frac{1}{\gamma}}}{1 + (\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{-\frac{1}{\gamma}}} W_{T-1}$$

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1}, T-1) = \delta^{T-1} \frac{C_{T-1}^{1-\gamma}}{1-\gamma} + \delta^{T} E_{T-1}\left[\frac{\left(R^{*}_{T-1}(W_{T-1} - C^{*}_{T-1})\right)^{1-\gamma}}{1-\gamma} \right]$$

Substitute C_{T-1} from 1.1)

$$= \delta^{T-1} \left(\frac{a_1}{1+a_1}\right)^{1-\gamma} \frac{W_{T-1}}{1-\gamma} + \delta^{T} E_{T-1}\left[\frac{R_{T-1}^{(1-\gamma)}}{1-\gamma} \left(W_{T-1} - \frac{a_1}{1+a_1} \cdot W_{T-1} \right)^{1-\gamma} \right]$$

$$= \delta^{T-1} \left(\frac{a_1}{1+a_1}\right)^{1-\gamma} \frac{W_{T-1}}{1-\gamma} + \delta^{T} E_{T-1}\left[R_{T-1}^{1-\gamma} \cdot \frac{W_{T-1}^{(1-\gamma)}}{(1-\gamma)(1+a_1)^{1-\gamma}} \right]$$

$$= \delta^{T-1} \cdot \frac{W_{T-1}^{1-\gamma}}{1-\gamma(1+a_1)^{1-\gamma}} \cdot \left(a_1^{1-\gamma} + \delta E_{T-1}\left[R_{T-1}^{(1-\gamma)}\right]\right)$$

let $b_1 = a^{1-\gamma} + \delta E\left[R_{T-1}^{(1-\gamma)}\right]) / (1+a_1)^{1-\gamma}$

$$= a_1 a_1^{-\gamma} + a_1^{-\gamma} / (1+a_1)^{1-\gamma} = \frac{(1+a_1)^{\gamma}}{a_1^{\gamma}}$$

$$= \left(\frac{a_1}{1+a_1}\right)^{-\gamma}$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

let $b_1 = (a_1^{1-\gamma} + \delta E_{T-1}[R_{T-1}^{(1-\gamma)}]) / (1+a_1)^{1-\gamma}$

$$U_c(C^*_{T-2}, T-2) = E_{T-2}[J_W(W_{T-1}, T-1) R_{T-2}]$$

$$= \delta^{T-1} E_{T-2}[b_1 W_{T-1}^{-\gamma} R_{T-2}]$$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}[b_1 (S_{T-2} R_{T-2})^{-\gamma} R_{T-2}]$$

$$= \delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}](W_{T-2} - C_{T-2})^{-\gamma}$$

so $C_{T-2} = \dfrac{(\delta b_1 E_{T-2}(R_{T-2}^{-\gamma}))^{-\frac{1}{\gamma}}}{1 + (\delta b_1 E_{T-1}[R_{T-2}^{1-\gamma}])^{-1/\gamma}} W_{T-2}$

$$E_{T-2}[b_1 (S_{T-2} R_{T-2})^{-\gamma} R_{r,T-2}] = R_f E_{T-2}[b_1 (S_{T-2} R_{T-2})^{-\gamma}]$$

$$E[R_{T-2}^{-\gamma} R_{r,T-2}] = R_f E[R_{T-2}^{-\gamma}]$$

$W^*_{T-2} = W^*_{T-1}$, not depend on wealth or C

as I mention in 1.1)

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

let $a_2 = \left(b_1 \delta E[R_{T-2}^{-\gamma}] \right)^{-\frac{1}{\gamma}}$

$$J(W_{T-2}, T-2) = U(C^*_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)]$$

$$= \delta^{T-2} \frac{C^{1-\gamma}}{1-\gamma} + E_{T-1}\left[\delta^{T-1} b_1 W_{T-2}^{1-\gamma} / 1-\gamma\right]$$

$$= \frac{\delta^{T-2}}{1-\gamma}\left(\frac{a_2}{1+a_2}\right)^{1-\gamma} W_{T-2}^{1-\gamma} + \frac{\delta^{T-1}}{1-\gamma} E_{T-1}\left[b_1 R_{T-2}^{1-\gamma} \frac{W_{T-2}^{1-\gamma}}{(1+a_2)^{1-\gamma}}\right]$$

$$= \frac{\delta^{T-2}}{1-\gamma} \cdot W_{T-2}^{1-\gamma} \cdot \underbrace{\frac{(a_2^{1-\gamma} + \delta E_{T-2}[b_1 R_{T-2}^{(1-\gamma)}])}{1+a_2}}_{b_2}$$

$$= \frac{\delta^{T-1} b_2 W_{T-2}^{1-\gamma}}{1-\gamma}$$

in T-1 , $C^*_{T-2} = c_2 W_{T-2}$ when $c_2 = \frac{a_1 c_1}{1+a_1 c_1}$

or $C_t = a_1 C_{t-1} / (1+a_i C_{t-1})$