Exercise 10

Differentiation of function with several variables: Partial derivatives/ Chain Rules

1. Calculate the partial derivatives (i.e. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$) of the following functions.

   (a) $f(x, y) = 5y^5 + 4x^4y + 3x^2y^3 + 2xy^4 + 2x + 3y + 5$

   Solution

   $\dfrac{\partial f}{\partial x} = 16x^3y + 6xy^3 + 2y^4 + 2$

   $\dfrac{\partial f}{\partial y} = 25y^4 + 4x^4 + 9x^2y^2 + 8xy^3 + 3$

   (b) $f(x, y) = \dfrac{xy^2}{x^2y^3 + 1}$

   Solution

   $\dfrac{\partial f}{\partial x} = \dfrac{y^2(x^2y^3 + 1) - xy^2(2xy^3)}{(x^2y^3 + 1)^2} = \dfrac{y^2(1 - x^2y^2)}{(x^2y^3 + 1)^2}$

   $\dfrac{\partial f}{\partial y} = \dfrac{2xy(x^2y^3 + 1) - xy^2(3x^2y^2)}{(x^2y^3 + 1)^2} = \dfrac{xy(2 - x^2y^2)}{(x^2y^3 + 1)^2}$

   (c) $f(x, y) = (x^9y + 1)(xy^8 + 1)$

   Solution

   $\dfrac{\partial f}{\partial x} 9x^8y(xy^8 + 1) + y^8(x^9y + 1)$

   $\dfrac{\partial f}{\partial y} = x^9(xy^8 + 1) + 8xy^7(x^9y + 1)$

2. Find all second derivatives (i.e. $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, , $\frac{\partial^2 z}{\partial x \partial y}$, and $\frac{\partial^2 z}{\partial y^2}$) of

$$z = 5x^2y + 3x^2y^2 + 5y^3.$$

   Solution

   $\dfrac{\partial z}{\partial x} = 10xy + 6xy^2$

   $\dfrac{\partial z}{\partial y} = 5x^2 + 6x^2y + 15y^2$

   $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(10xy + 6xy^2\right) = 10y + 6y^2$

   $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(10xy + 6xy^2\right) = 10x + 12xy$

   $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(5x^2 + 6x^2y + 15y^2\right) = 10x + 12xy$

   $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(5x^2 + 6x^2y + 15y^2\right) = 6x^2 + 30y$

3. Find $\frac{\partial z}{\partial x}$ at (0,1) for $z = e^{-x}\cos(y)$. Ans: $\frac{\partial z}{\partial x} = -\cos(1)$

4. Answer the following questions.
   (a) Express the volume of a right circular cylinder as a function of two variables: its radius r and its height h.
   (b) Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.
   (c) Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

   Ans: (a) $V(r,h) = \pi r^2 h$, (b) $\frac{\partial V}{\partial r} = 2\pi rh$, (c) $\frac{\partial z}{\partial h} = \pi r^2$.

5. Find f_{xy} for $z = \ln(x-y)$. Ans: $f_{xy} = \frac{1}{(x-y)^2}$

6. Given $f(x,y,z) = xyz$, find f_{xyy}, f_{xyx}, and f_{yyx}. Ans: $f_{xyy} = f_{xyx} = f_{yyx} = 0$

7. Given $f(x,y) = x^2 + x - 3xy + y^3 - 5$, find all points at which $f_x = f_y = 0$ simultaneously.
   Ans: $(x,y) = (\frac{1}{4}, \frac{1}{2}), (1,1)$

8. Show that $z = e^x \sin(y)$ satisfies the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.
   Ans: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin(y) - e^x sin(y) = 0$

9. A Cobb-Douglas production function is $f(x,y) = 200x^{0.7}y^{0.3}$, where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000$. Fin $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at these values, which represent the marginal productivity of labor and capital, respectively.
   Ans: $\frac{\partial f}{\partial x}$ at $(500, 1000) = 172.36$, $\frac{\partial f}{\partial y}$ at $(500, 1000) = 36.93$.

10. Let $w(x,y,z) = xy\cos(z)$, where $x = t$, $y = t^2$, and $z = \arcsin(t)$. Find $\dfrac{dw}{dt}$. (the answer can be given in terms of x, y, z and t)
    Ans: $\dfrac{dw}{dt} = y\cos(z) + 2tx\cos(z) - \dfrac{xy\sin(z)}{\sqrt{1-t^2}}$

11. If $w = 5x^2 + 2y^2$, $x = -3s + t$, and $y = s - 4t$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ by using the chain rule.
    Ans: $\frac{\partial w}{\partial s} = -30x + 4y$ and $\frac{\partial w}{\partial t} = 10x - 16y$

12. If $f(x,y) = xy$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$, find $\frac{\partial f}{\partial r}$ and express the answer in terms of r and θ.
    Ans: $\frac{\partial f}{\partial r} = r\sin(2\theta)$ (note: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$)

13. For the following exercises, find $\dfrac{dy}{dx}$ using partial derivatives (i.e. using the theorem on implicit differentiation).
    (a) $x^2 - 2xy + y^4 = 4$ Ans: $\dfrac{dy}{dx} = \dfrac{y-x}{-x+2y^3}$
    (b) $e^{xy} + ye^y = 1$ Ans: $\dfrac{dy}{dx} = -\dfrac{ye^{xy}}{xe^{xy} + e^y(1+y)}$

14. If $z = xye^{x/y}$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ when $r = 2$ and $\theta = \dfrac{\pi}{6}$.
Ans: $\dfrac{\partial z}{\partial r} = \sqrt{3}e^{\sqrt{3}}$, $\dfrac{\partial z}{\partial \theta} = (2 - 4\sqrt{3})e^{\sqrt{3}}$,

15. Let $u = u(x, y, z)$, where $x = x(w, t)$, $y = y(w, t)$, $z = z(w, t)$, $w = w(r, s)$, and $t = t(r, s)$. Use a tree diagram and the chain rule to find an expression for $\dfrac{\partial u}{\partial r}$.
Ans: $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\left(\dfrac{\partial x}{\partial w}\dfrac{\partial w}{\partial r} + \dfrac{\partial x}{\partial t}\dfrac{\partial t}{\partial r}\right) + \dfrac{\partial u}{\partial y}\left(\dfrac{\partial y}{\partial w}\dfrac{\partial w}{\partial r} + \dfrac{\partial y}{\partial t}\dfrac{\partial t}{\partial r}\right) + \dfrac{\partial u}{\partial z}\left(\dfrac{\partial z}{\partial w}\dfrac{\partial w}{\partial r} + \dfrac{\partial z}{\partial t}\dfrac{\partial t}{\partial r}\right)$

16. For the following exercises, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. (Hint: use the theorem on implicit differentiation)

   (a) $x^3z^2 - 5xy^5z = x^2 + y^3$
   Ans: $\frac{\partial z}{\partial x} = \frac{2x-3x^2z^2+5y^5z}{2x^3z-5xy^5}$, $\frac{\partial z}{\partial y} = \frac{3y^2+25xy^4z}{2x^3z-5xy^5}$

   (b) $x^2\sin(2y - 5z) = 1 + y\cos(6zx)$
   Ans: $\frac{\partial z}{\partial x} = \frac{2x\sin(2y-5z)+6zy\sin(6zx)}{5x^2\cos(2y-5z)-6yx\sin(6zx)}$, $\frac{\partial z}{\partial y} = \frac{\cos(6zx)-2x^2\cos(2y-5z)}{6xy\sin(6zx)-5x^2\cos(2y-5z)}$