Chapter 9

Optimization with Equality Constraint with many choice variables

- Lagrange multiplier
- Conditions for optimization
- Maximize output level subject to cost constraint
- Minimize cost subject to output constraint
- ~~Minimize~~ maximize utility subject to budget constraint
- Minimize total expenditure under the level of utility

✽

Transition of idea

Ch. 8	Ch. 9
Unconstrained optimization	Constrained optimization
max/min $z = f(x_1, x_2)$	max/min $z = f(x_1, x_2)$
x_1, x_2	x_1, x_2
	s.t. $h(x_1, x_2) = c$ ⇒ constraint
	Lagrangian function
	max/min $L = f(x_1, x_2) + \lambda[c - h(x_1, x_2)]$
	x_1, x_2, λ
	as if λ becomes another choice variable

FOC partial derivative w.r.t. every choice variables

SOC matrix that collects every possible 2nd order partial derivative of obj fn / Lagrangian fn

Hessian Matrix | Bordered Hessian matrix

What if choice variables cannot be chosen freely? For example, what if a firm cannot freely chooses quantities because of production quota? Firm will need to find constrained optimum value.

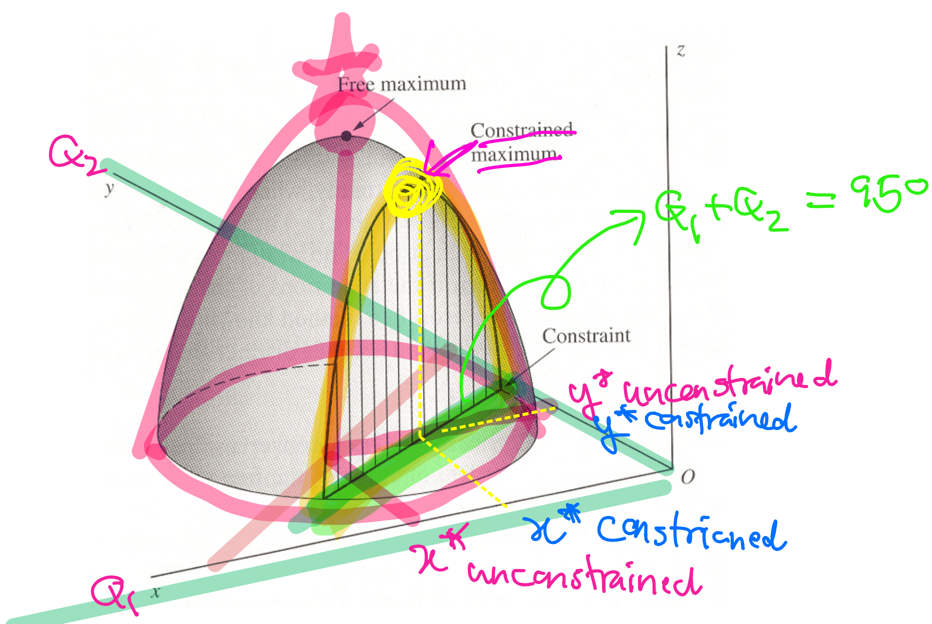
Suppose that a firm has two factories and is subjected to production quota of 950. The profit maximization problem becomes:

$$\max_{Q_1, Q_2} \pi(Q_1, Q_2)$$

$$\text{Subject to } Q_1 + Q_2 = 950$$

,where Q_1, Q_2 are quantity produced at factory 1 and 2, respectively.

Free Extremum vs. Constrained Extremum



Economics is about finding the optimal allocation of scarce resources. Hence, the main problems in economics can be studied as constrained optimization problem. The prototype of such problem is:

$$\max_{x_1, x_2, \ldots, x_n} imize\ f(x_1, x_2 \ldots, x_n)$$
$$where(x_1, x_2, \ldots, x_n)\ must\ satisfy$$

$$\left.\begin{matrix} g_1(x_1, x_2, \ldots, x_n) \le b_1 \\ g_2(x_1, x_2, \ldots, x_n) \le b_2 \\ \ldots\ldots\ldots\ldots \\ g_n(x_1, x_2, \ldots, x_n) \le b_n \end{matrix}\right\} \text{inequality constraint}$$

$$\left.\begin{matrix} h_1(x_1, x_2, \ldots, x_n) = c_1 \\ h_2(x_1, x_2, \ldots, x_n) = c_2 \\ \ldots\ldots\ldots\ldots \\ h_n(x_1, x_2, \ldots, x_n) = c_n \end{matrix}\right\} \text{equality constraint}$$

What will be focusing on?

✽

Optimization with Equality Constraints :Two variables and One Equality Constraint

Let Mr. Musk have utility function $U = x_1x_2 + 2x_1$. If $P_1 = 4$ and $P_2 = 2$ Baht per piece, how many piece of good 1 and good 2 will Mr. Musk choose to consume to maximize his utility?

Suppose that Mr. Musk has no budget constraint

$$\max_{x_1, x_2} U = x_1x_2 + 2x_1$$

FOC:

$U_{x_1} = x_2 + 2 = 0 \rightarrow x_2^* = -2$

$U_{x_2} = x_1 = 0 \rightarrow x_1^* = 0$

SOC:

$$H = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

H_1, $H_2 = H$

$|H_1| = |0| = 0 \ \ngtr 0, \nless 0$

$|H_2| = -1 \ \ngtr 0$

H is indefinite

x_1^*, x_2^* don't give U_{max}

$U_{x_1} = MU_{x_1} = x_2 + 2 \Rightarrow \frac{\partial MU_{x_1}}{\partial x_1} = 0$

$U_{x_2} = MU_{x_2} = x_1 \Rightarrow \frac{\partial MU_{x_2}}{\partial x_2} = 0$

Law of diminishing MU doesn't apply here.

Now suppose that Mr. Musk has the budget equal to 60 baht, how many piece of good 1 and good 2 will Mr. Musk choose to consume to maximize his utility?

Mr. Musk's budget const is $4x_1 + 2x_2 = 60$

$2x_2 = 60 - 4x_1$

$\hookrightarrow x_2 = 30 - 2x_1$

$\max_{x_1} U = x_1(30 - 2x_1) + 2x_1$

"Lagrange Multiplier"

Let f and h be functions with two independent variables x_1, x_2

$$\boxed{\begin{aligned}&\max_{x_1,x_2} f(x_1,x_2)\\ &\text{Subject to } h(x_1,x_2) = c\end{aligned}} \rightarrow c - h(x_1,x_2) = 0$$

"Lagrangian Function"

$$\max_{x_1,x_2,\mu} \boxed{L(x_1,x_2,\mu) = f(x_1,x_2) + \mu\left[c - h(x_1,x_2)\right]}$$

lagrange multiplier

We will call μ Lagrange multiplier.

Treat μ as if it is another choice variable.

When the constraint is satisfied, $c - h(x_1,x_2) = 0$,

maximizing/minimizing lagrangian f^n is equivalent to

" ______ " the objective f^n

FONC: choice vars are x_1, x_2, μ

$$\left.\begin{aligned}\frac{\partial L}{\partial x_1} &= f_1 - \mu h_1(x_1,x_2) = 0\\ \frac{\partial L}{\partial x_2} &= f_2 - \mu h_2(x_1,x_2) = 0\end{aligned}\right\} \text{w.r.t. every choice var}$$

$$\left.\frac{\partial L}{\partial \mu} = \underbrace{c - h(x_1,x_2) = 0}_{h(x_1,x_2) = c \text{ "the constraint"}}\right\} \text{w.r.t. } \mu$$

solve for x_1^*, x_2^*, μ^* that satisfy all FOCs.

Mr. Musk's problem is

$$\max_{x_1,x_2} U = \overbrace{x_1x_2 + 2x_1}^{f(x_1,x_2)}$$

$$\text{st. } \underbrace{4x_1 + 2x_2}_{h(x_1,x_2)} = \underset{c}{60}$$

$60 - 4x_1 - 2x_2 = 0$

Lagrangian function for utility maximization problem when Mr.Musk has 60 baht is:

$$\max_{x_1, x_2, \mu} L = x_1x_2 + 2x_1 + \mu[60 - 4x_1 - 2x_2]$$

these are there because of we consider info. from the contraint.

FONC:

$L_\mu = \frac{\partial L}{\partial \mu} = 60 - 4x_1 - 2x_2 = 0 \Rightarrow 4x_1 + 2x_2 = 60 \quad (1)$

$L_1 = \frac{\partial L}{\partial x_1} = x_2 + 2 - 4\mu = 0 \Rightarrow x_2 + 2 = 4\mu \quad (2)$

$L_2 = \frac{\partial L}{\partial x_2} = x_1 - 2\mu = 0 \Rightarrow x_1 = 2\mu \quad (3.)$

(2) / (3.) $\quad \frac{x_2+2}{x_1} = \frac{4\mu}{2\mu} = 2 \Rightarrow x_1 = \frac{x_2+2}{2} \quad (4.)$

(4.) in (1.) $\quad 4\left(\frac{x_2+2}{2}\right) + 2x_2 = 60$

$x_2^* = 14$

$x_1^* = 8 \quad , \quad \mu^* = \frac{x_1^*}{2} = 4$

Homework: Write Lagrangian function, first order conditions, and critical values of the following problems:

(a.)

$Max\ U(x_1, x_2) = x_1x_2$

$St. \quad h(x_1, x_2) = x_1 + 4x_2 = 16$

(b.)

$Max\ z = xy$

$St. \quad x + y = 6$

✿

Optimization with Equality Constraints : n variables and One Equality Constraint

$$\max_{x_1,\dots,x_n} \quad f(x_1, x_2, \dots, x_n)$$

$$st. \; h(x_1, x_2, \dots, x_n) = c$$

Lagrangian function is

$$\max_{x_1,\dots,x_n,\mu} \quad L = f(x_1,\dots,x_n) + \mu\big[\underbrace{c - h(x_1,\dots,x_n)}_{=0 \text{ if the const. is satisfied.}}\big]$$

FOC :

$$L_\mu = \frac{\partial L}{\partial \mu} = c - h(x_1,\dots,x_n) = 0$$

$$L_1 = \frac{\partial L}{\partial x_1} = f_1 - \mu h_1 = 0$$

$$L_2 = \frac{\partial L}{\partial x_2} = f_2 - \mu h_2 = 0$$

$$\vdots$$

$$L_n = \frac{\partial L}{\partial x_n}$$

Use n+1 FOCs to solve for critical value of $x_1^*, x_2^*, \dots, x_n^*, \mu^*$

Note 1 : At critical values, the constraint is satisfied,

$$\boxed{L^* = f(x_1^*, x_2^*, \dots, x_n^*) + \mu^*\big[\underbrace{c - h(x_1^*, x_2^*, \dots, x_n^*)}_{0}\big]}$$

$\mu^* \neq 0$

$$L^* = f(x_1^*, x_2^*, \dots, x_n^*)$$

The value of lagrangian fn is equal to the value of obj fn at the critical point.

Note 2 : $\dfrac{\partial L^*}{\partial c} = \mu^* = \dfrac{\partial f(x_1^*,\dots,x_n^*)}{\partial c}$

shows how optimized value of objective fn changes when constraint c changes 1 unit

ex. when budget increases by 1 baht, how much will the maximized utility change? ⇒ it would change by μ^* utils.

Second-Order Sufficient Condition

Consider objective function: $\max\limits_{x_1,\dots,x_n} f(x_1, x_2, \dots, x_n)$

$$
\begin{aligned}
St.\quad & h^1(x_1, x_2, \dots, x_n) = c_1 \\
& h^2(x_1, x_2, \dots, x_n) = c_2 \\
& \dots\dots\dots\dots\dots\dots \\
& h^k(x_1, x_2, \dots, x_n) = c_k
\end{aligned}
\Bigg\} \text{ } k \text{ equality consts.}
$$

k lagrange multipliers.

The Lagrangian

max $\left.\begin{matrix} x_1,\dots,x_n, \\ \mu_1,\dots,\mu_k \end{matrix}\right\} n+k$

n choice vars, k lagrange multipliers

$$L(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + \underbrace{\mu_1\left[c_1 - h^1(x_1, x_2, \dots, x_n)\right] + \mu_2\left[c_2 - h^2(x_1, x_2, \dots, x_n)\right] + \dots + \mu_k\left[c_k - h^k(x_1, x_2, \dots, x_n)\right]}_{k \text{ terms}}$$

<u>First Order Necessary Condition: FONC</u>

$n+k$ eqs.

$$
\left.
\begin{aligned}
L_{x_1} &= \frac{\partial L}{\partial x_1} = 0 \quad f_1 - \mu_1 h^1_1 - \mu_2 h^2_1 - \dots - \mu_k h^k_1 = 0 \\
&\vdots \\
L_{x_n} &= \frac{\partial L}{\partial x_n} = 0 \quad f_n - \mu_1 h^1_n - \mu_2 h^2_n - \dots - \mu_k h^k_n = 0
\end{aligned}
\right\} n \text{ eq.}
$$

$$
\left.
\begin{aligned}
L_{\mu_1} &= \frac{\partial L}{\partial \mu_1} = 0 \quad c_1 - h^1(x_1,\dots,x_n) = 0 \\
&\vdots \\
L_{\mu_k} &= \frac{\partial L}{\partial \mu_k} = 0 \quad c_k - h^k(x_1,\dots,x_n) = 0
\end{aligned}
\right\} k \text{ eqs.}
$$

<u>Second Order Sufficient Condition: SOSC</u>

"Bordered Hessian Matrix"……… $\bar{H}$: a matrix collecting every possible 2^{nd} order partial derivatives of the lagrangian f^n

1^{st} order partial derivatives : $L_{\mu_1}, L_{\mu_2}, \dots, L_{\mu_k}, L_{x_1}, L_{x_2}, \dots, L_{x_n}$: $n+k$ terms.

Take partial diff of 1^{st} order w.r.t. : $\mu_1, \dots, \mu_k, x_1, \dots, x_n$

We can divide Bordered Hessian Matrix into 4 parts:

Note: We evaluate Bordered Hessian at the critical values.

partial diff w.r.t.

$$
\begin{array}{r|cccccccc|l}
 & \text{Col }1 & \text{Col }2 & & \text{Col }k & \text{Col }k+1 & \text{Col }k+2 & & \text{Col }k+n & \\
 & L_{\mu_1} & L_{\mu_2} & \cdots & L_{\mu_k} & L_{x_1} & L_{x_2} & \cdots & L_{x_n} & \\
\hline
\text{row }1 & L_{\mu_1\mu_1} & L_{\mu_1\mu_2} & \cdots & L_{\mu_1\mu_k} & L_{\mu_1 x_1} & L_{\mu_1 x_2} & \cdots & L_{\mu_1 x_n} & \mu_1 \\
\text{row }2 & L_{\mu_2\mu_1} & L_{\mu_2\mu_2} & \cdots & L_{\mu_2\mu_k} & L_{\mu_2 x_1} & L_{\mu_1 x_2} & \cdots & L_{\mu_1 x_n} & \mu_2 \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\text{row }k & L_{\mu_k\mu_1} & L_{\mu_k\mu_2} & \cdots & L_{\mu_k\mu_k} & L_{\mu_k x_1} & L_{\mu_k x_2} & \cdots & L_{\mu_k x_n} & \mu_k \\
\text{row }k+1 & L_{x_1\mu_1} & L_{x_1\mu_2} & \cdots & L_{x_1\mu_k} & L_{11} & L_{12} & \cdots & L_{1n} & x_1 \\
 & L_{x_2\mu_1} & L_{x_2\mu_2} & \cdots & L_{x_2\mu_k} & L_{21} & L_{22} & \cdots & L_{2n} & x_2 \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
\text{row }k+n & L_{x_n\mu_1} & L_{x_n\mu_2} & \cdots & L_{x_n\mu_k} & L_{n1} & L_{n2} & \cdots & L_{nn} & x_n
\end{array}_{(n+k)\times(n+k)}
$$

$\bar{H} =$ the matrix above. H; Border; A; k×n; A'; n×k; Hessian

$L_{\mu_i\mu_j} = 0 \quad i=1,\dots,k \quad j=1,\dots,k$

taking partial diff of the first const w.r.t. $x_1,\dots,x_n$ and with negative sign

$$
\bar{H} = \begin{bmatrix}
0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\
0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\
-h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\
-h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn}
\end{bmatrix}_{(n+k)\times(n+k)}
$$

We can have submatrix H_1 with border, $\bar{H}_1$, and so on. Try to get the submatrix from $\bar{H}$.

$\bar{H}_1$ is H_1 including all rows above & all column to the left

$\bar{H}_1$

$$
\bar{H}_1 = \begin{bmatrix}
0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\
0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\
-h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\
-h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn}
\end{bmatrix}_{(n+k)\times(n+k)}
$$

H_1 $(k+1)\times(k+1)$

$\bar{H}_2$

$$
\bar{H}_2 = \begin{bmatrix}
0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\
0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\
-h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\
-h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn}
\end{bmatrix}_{(n+k)\times(n+k)}
$$

H_2 $(k+2)\times(k+2)$

Question:
How many submatrices do we need to consider?

Answer:
$n-k$ matrices, i.e. the last n-k leading principle minors of matrix $\bar{H}$.

$n > k$

n is the number of choice variables and k is the number of constraints.

We need to focus since $\bar{H}_{k+1}$.

$$\underbrace{\bar{H}_1, \bar{H}_2, \bar{H}_3, \dots \bar{H}_k}_{\text{drop off from our consideration}}, \underbrace{\bar{H}_{k+1}, \dots, \bar{H}_n}_{\text{consider the last } n-k \text{ submatrices}}$$

We need to check:

$$\bar{H}_{k+1}, \ \bar{H}_{k+2}, \dots\dots \bar{H}_n = \bar{H}$$

Example 1: the case of one constraint with two choice variables:

$$
\bar{H} = \begin{array}{cc}
 & \begin{array}{ccc} L_{\mu\mu} & L_{\mu x_1} & L_{\mu x_2} \end{array} \\
\begin{array}{c} \\ L_{x_1\mu} \\ L_{x_2\mu} \end{array} &
\begin{bmatrix}
0 & -h_{x_1} & -h_{x_2} \\
-h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\
-h_{x_2} & L_{x_2x_1} & L_{x_2x_2}
\end{bmatrix}
\end{array}
$$

3×3
$(2+1)\times(2+1)$

Number of choice variables (n) ……2…..

Number of constraint (k) ……1…..

So we need to check the last n-k = ……2 − 1 = 1……..Matrix, which is:

k=1

$$\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} = \bar{H}$$

Example 2: 3 choice variables and 1 constraint

$\bar{H}_1$ $\bar{H}_2$ $\bar{H}_3$

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ -h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} (3+1)\times(3+1)$$

$\cancel{\bar{H}_1}$ drop

$\boxed{\bar{H}_2 \quad \bar{H}_3}$ keep

Number of choice variables (n) …3……..

Number of constraint (k) …1…..

keep the last n-k, drop the first k

So we need to check the last n-k = ……3 − 1 = 2……..Matrix, beginning at $\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2$

$$\bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} \qquad \bar{H}_3 = \bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ -h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix}$$

We need to observe pattern of the determinants of leading principal submatrices as the following.

The objective function will be at the local max if the last (n-k) leading principal minors have *alternate signs with* $|\bar{H}_{k+1}|$ has the same sign as $(-1)^{k+1}$.

The objective function will be at the local min if the last (n-k) leading principal minors have the *same sign* as $(-1)^k$ for every bordered Hessian that we need to consider.

ↀ SOC of 2 choice variables + 1 equality constraint ↀ

$$|\bar{H}_2| = \begin{vmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{vmatrix}$$ 3x3

If $|\bar{H}_2| > 0$ ➔ minimized

If $|\bar{H}_2| < 0$ ➔

$n = 2$, $k = 1$ (consider x_1, x_2, μ when taking partial diff of lagrangian f^n)

look at the last $n-k = 2-1 = 1$ matrix

which is $\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2$

Obj. f^n is maximized when $|\bar{H}_{k+1}|$ has the same sign as $(-1)^{k+1} = (-1)^{1+1} = (-1)^2 = 1 > 0$

$\therefore |\bar{H}_{k+1}| > 0$

$|\bar{H}_2|$

has the same sign as $(-1)^k = (-1)^1 = -1 < 0$

$|\bar{H}_{k+1}| < 0$

$" \; |\bar{H}_2|$

☙ FOC, SOC ของ n choice variables + 1 equality constraint ❧

$$\max_{x_1, x_2, \ldots, x_n} / \min \quad f(x_1, x_2, \ldots, x_n)$$

$$St. \quad \cancel{g}(x_1, x_2, \ldots, x_n) = c \qquad h(x_1, \ldots, x_n) = c$$

$$\max_{x_1, \ldots, x_n, \lambda} / \min \quad L = f(x_1, x_2, \ldots, x_n) + \lambda[c - g(x_1, x_2, \ldots, x_n)]$$

FOC:

$$L_1 = L_2 = L_3 = \ldots = L_n = L_\lambda = 0$$

SOC: look at the last n-k = n-1 leading submatrices, starting to look at $\bar{H}_{k+1} = \bar{H}_2$

✪ **MAX**

$$|\bar{H}_2| > 0 \,;\ |\bar{H}_3| < 0 \,; |\bar{H}_4| > 0 \,; \ldots\ldots\ldots$$

$|\bar{H}_2|$ has the same sign as $(-1)^{k+1} = (-1)^{1+1} > 0$ and alternate

✪ **MIN**

$$|\bar{H}_2|;\ |\bar{H}_3|; |\bar{H}_4|; \ldots; |\bar{H}_n| < 0$$

$|\bar{H}_2|$ has the same sign as $(-1)^k = (-1)^1 < 0$

$\bar{H}_2$ $\quad$ $\bar{H}_3$ $\quad$ $\bar{H}_n = \bar{H}$

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} & \cdots & -h_{x_n} \\ -h_{x_1} & L_{11} & L_{12} & L_{13} & \cdots & L_{1n} \\ -h_{x_2} & L_{21} & L_{22} & L_{23} & \cdots & L_{2n} \\ -h_{x_3} & L_{31} & L_{32} & L_{33} & \cdots & L_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_{x_n} & L_{n1} & L_{n2} & L_{n3} & \cdots & L_{nn} \end{bmatrix} \quad (n+1)\times(n+1)$$

firm's

$\max_{K,L} Q$ st. C

Output maximization problem

$\min_{K,L} TC$ st. Q_0

Cost minimization problem (CMP)

Consumer's

$\max_{x,y} U$ st. B

Utility Maximization Problem (UMP)

$\min_{x,y} Exp$ st. U_0

Expenditure minimization Problem (EMP)

✽ Maximizing output level subject to cost constraint

Let the production function be $Q = q(L, K)$

The total cost is:

$$TC = wL + rK = C$$

Question: How will firm choose L and K to maximize output level, given that the total cost must be C baht?

$$\max_{L,K} Q = q(L,K)$$
$$\text{st.} \quad wL + rK = C$$

Step 1: Lagrangian Function

$$\max_{\lambda, L, K} Z = q(L,K) + \lambda[C - wL - rK]$$

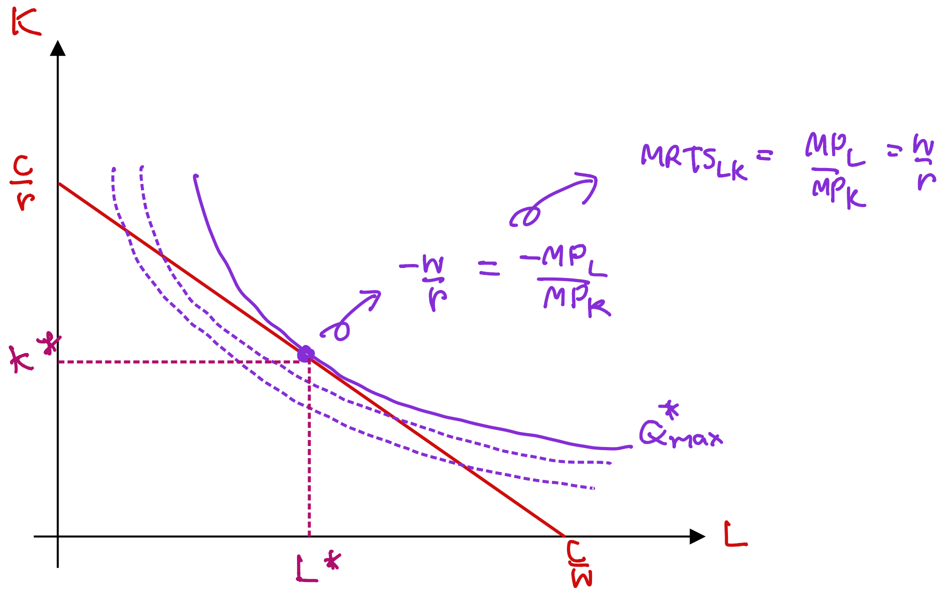
Step 2: First-Order Condition (FONC):

$$\frac{\partial Z}{\partial \lambda} = C - wL - rK = 0 \Rightarrow wL + rK = C \quad \text{---(1)}$$
$$\frac{\partial Z}{\partial L} = \frac{\partial q(L,K)}{\partial L} - w\lambda = 0 \Rightarrow q_L = w\lambda \quad \text{---(2)}$$
$$\frac{\partial Z}{\partial K} = \frac{\partial q(L,K)}{\partial K} - r\lambda = 0 \Rightarrow q_K = r\lambda \quad \text{---(3)}$$

use (1.), (2.), (3.) to solve for K^*, L^*, λ^*

$$(2.)/(3.) \quad MRTS_{LK} = \frac{q_L}{q_K} = \frac{w}{r}$$

Economic Interpretation



isocost : $wL + rK = C$ $\quad\therefore\ rK = C - wL$

$\Rightarrow K = \frac{C}{r} - \frac{w}{r}L$

isoquant : $Q = q(L,K) = \bar{Q}$

$dQ = q_L\, dL + q_K\, dK = d\bar{Q} = 0$

$$\boxed{\frac{dK}{dL} = -\frac{q_L}{q_K}}$$

note : $\frac{d^2K}{dL^2} > 0$: $\boxed{\frac{d^2K}{dL^2} = \frac{\partial}{\partial L}\left(\frac{-q_L}{q_K}\right)\frac{dL}{dL} + \frac{\partial}{\partial K}\left(-\frac{q_L}{q_K}\right)\frac{dK}{dL}} > 0$

note 2 : $z^* = q(L^*,K^*) + \lambda^*[\underbrace{C - wL^* - rK^*}_{=0}]$ $\searrow$ $z^* = q(L^*,K^*)$

$$\frac{\partial z^*}{\partial C} = \lambda^* = \frac{\partial q(L^*,K^*)}{\partial C}$$

$$= \frac{1}{\frac{\partial C}{\partial Q^*}} = \frac{1}{MC^*}$$

$C\uparrow$ 1 baht $\rightarrow$ $Q^*_{max}\uparrow$ $\lambda^* = \frac{1}{MC^*}$ units.

The number of the maximized output that can be increased when budget for cost increases 1 bath.

$\frac{\partial Z}{\partial \lambda} = C - wL - rK$

$\frac{\partial Z}{\partial L} = \frac{\partial q(L,K)}{\partial L} - w\lambda$

$\frac{\partial Z}{\partial K} = \frac{\partial q(L,K)}{\partial K} - r\lambda$

Step 3: Second-Order Sufficient Condition

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} z_{\lambda\lambda} = 0 & z_{\lambda L} = -w & z_{\lambda K} = -r \\ z_{L\lambda} = -w & z_{LL} = q_{LL} & z_{LK} = q_{LK} \\ z_{K\lambda} = -r & z_{KL} = q_{KL} & z_{KK} = q_{KK} \end{bmatrix}_{(n+k)\times(n+k)}$$

$\bar{H}_1$ $\quad \bar{H}_2 = \bar{H}$

$$\bar{H} = \begin{bmatrix} 0 & -w & -r \\ -w & q_{LL} & q_{LK} \\ -r & q_{KL} & q_{KK} \end{bmatrix}_{3\times 3}$$

3.2 Check the last n-k leading principle Minor

n = 2 (K, L) k = 1

We need to check the last $n-k = 2-1 = 1$ leading principle Minor, which is $|\bar{H}_{k+1}| = |\bar{H}_{1+1}| = |\bar{H}_2|$

needs to have the same sign as $(-1)^{k+1} = (-1)^{1+1} = 1 > 0$

$|\bar{H}_2| = |\bar{H}| > 0$

$$|\bar{H}| = wrq_{LK} + wrq_{KL} \overbrace{- r^2 q_{LL}}^{+} \overbrace{- w^2 q_{KK}}^{+}$$

$q_{LK} = q_{KL} > 0$

if $K\uparrow$, $MP_L \uparrow$

$L\uparrow$, $MP_K \uparrow$

$\frac{\partial MP_L}{\partial L} < 0$ $\quad \frac{\partial MP_K}{\partial K} < 0$

law of diminishing MP_L, MP_K

H.W. $\max_{L,K} \; Q = L^{0.4}K^{0.5}$ st. $3L + 4K = 108$

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Minimizing cost given a level of output:

Least Cost Combination of Inputs and Conditional Input Demand

Assume that a firm needs to produce output equal to Q_0 units from production function $Q = F(L, K)$, and the firm would like to choose the level of capital and labor that minimize the cost.

$$\left.\begin{aligned} \min_{L,K} \quad & TC = wL + rK \\ \text{st.} \quad & F(L,K) = Q_0 \end{aligned}\right\} \text{cost minimization problem (CMP)}$$

Step 1: **Lagrangian Function**

$$\min_{\lambda, L, K} \quad Z = wL + rK + \lambda\left[Q_0 - F(L,K)\right]$$

Step 2: **First-Order Condition (FONC):**

$$\frac{\partial Z}{\partial \lambda} = Q_0 - F(L,K) = 0 \Rightarrow \boxed{F(L,K) = Q_0} \quad (1.)$$

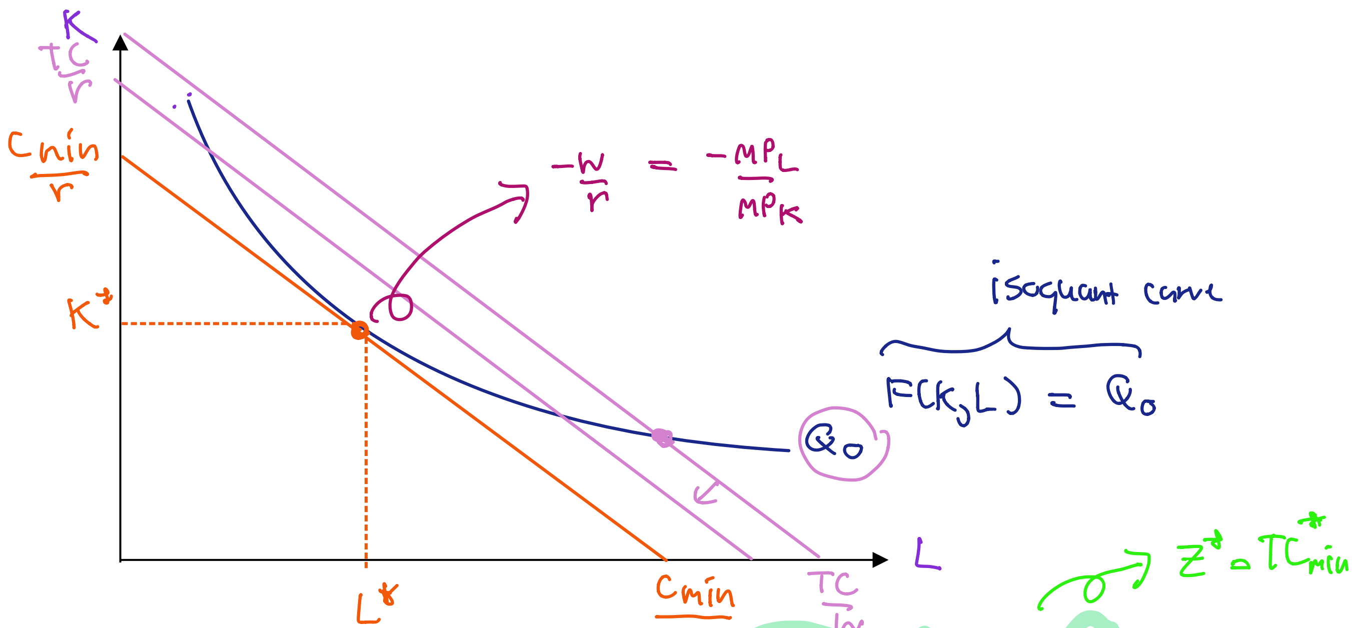
$$\frac{\partial Z}{\partial L} = w - \lambda \frac{\partial F(L,K)}{\partial L} = 0 \Rightarrow \lambda MP_L = w \quad (2.)$$

$$\frac{\partial Z}{\partial K} = r - \lambda \frac{\partial F(L,K)}{\partial K} = 0 \Rightarrow \lambda MP_K = r \quad (3.)$$

(2.)/(3.)

$$\boxed{\frac{MP_L}{MP_K} = \frac{w}{r}}$$

Economic Interpretation



$$Z^* = wL^* + rK^* + \lambda^* \underbrace{[Q_0 - F(L^*, K^*)]}_{=0}$$

$$\frac{\partial Z^*}{\partial Q_0} = \lambda^* = \frac{\partial TC^*_{min}}{\partial Q_0}$$

= marginal cost of production in the optimum state, $\lambda^* > 0$

Step 3 **Second-Order Sufficient Condition**

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} Z_{\lambda\lambda} = & Z_{\lambda L} = & Z_{\lambda K} = \\ Z_{L\lambda} = & Z_{LL} = & Z_{LK} = \\ Z_{K\lambda} = & Z_{KL} = & Z_{KK} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} 0 & -F_L & -F_K \\ -F_L & -\lambda F_{LL} & -\lambda F_{LK} \\ -F_K & -\lambda F_{KL} & -\lambda F_{KK} \end{bmatrix}$$

$(2+1)\times(2+1)$

3.2 Check the last n-k leading principle Minor

n = 2 k = 1

We need to check the last 2−1 = 1 leading principle Minor นั่นคือ $|\bar{H}_{k+1}| = |\bar{H}_2| = |\bar{H}|$

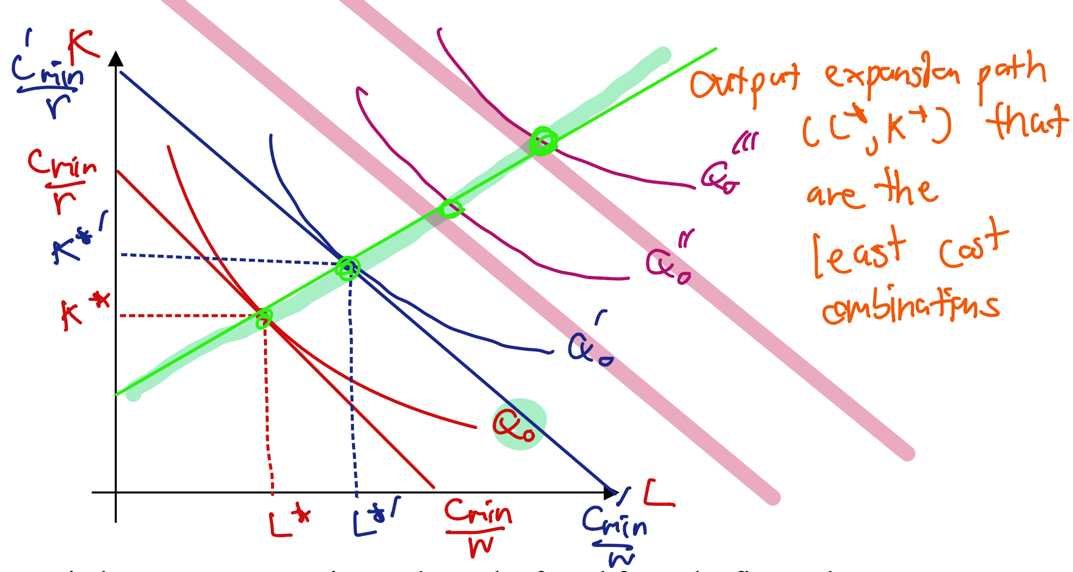
with the sign of $(-1)^k = (-1)^1 < 0$

$|\bar{H}_2| = -\lambda F_L F_K F_{LK} - \lambda F_L F_K F_{KL} + \lambda F_K^2 F_{LL} + \lambda F_L^2 F_{KK} < 0$

Given $\frac{w}{r}$ constant, what happen with K^*/L^* when we expand the production, increasing Q_0 ?

The Expansion Path

Expansion Path is the least cost combination between L* and K* given many levels of given outputs.



If Isoquants are strictly convex, expansion path can be found from the first order conditions.

If a firm has Cobb-Douglas production function $Q = AL^{\alpha}K^{\beta}$,

$$\boxed{MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{w}{r}}$$

$$\frac{\alpha A L^{\alpha-1} K^{\beta}}{\beta A L^{\alpha} K^{\beta-1}} = \frac{w}{r} \Rightarrow \frac{\alpha K^*}{\beta L^*} = \frac{w}{r} \quad \text{--- (4.)}$$

In this case, the optimal ratio of capital to labor is constant and is equal to $\boxed{\frac{K^*}{L^*} = \frac{\beta w}{\alpha r}}$

Output expansion path is $K^* = \frac{\beta w}{\alpha r} L^*$ ——— (5.)

linear expansion path for Cobb Douglas

$K^* = g(A, \alpha, \beta, Q_0)$

$L^* = f(A, \alpha, \beta, Q_0)$

Conditional factor demand vs. Factor demand

eq (1.) & Cobb-Douglas productⁿ fⁿ : $\boxed{A L^{\alpha} K^{\beta} = Q_0}$ — (1.)

substitute K out by using (5.) & putting in (1.)

$$A L^{\alpha} \left(\frac{\beta w}{\alpha r} L \right)^{\beta} = Q_0$$

$$L^{\alpha+\beta} = A^{-1} \left(\frac{\alpha r}{\beta w} \right)^{\beta} Q_0$$

$$L^* = A^{\frac{-1}{\alpha+\beta}} \left(\frac{\alpha r}{\beta w} \right)^{\frac{\beta}{\alpha+\beta}} Q_0^{\frac{1}{\alpha+\beta}} \quad ☆$$

" conditional labor demand "
conditional on Q_0

From $\frac{K^*}{L^*} = \frac{\beta w}{\alpha r} \Rightarrow L^* = \frac{\alpha r}{\beta w} K^*$ (6.)

use (6.) & (1.)

$$A \left(\frac{\alpha r}{\beta w} K \right)^{\alpha} K^{\beta} = Q_0$$

$$K^* = A^{-\frac{1}{\alpha+\beta}} \left(\frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} Q_0^{\frac{1}{\alpha+\beta}} \quad ☆☆$$

" conditional capital demand "
on Q_0

Cost minimization problem vs. Profit maximization problem

conditional factor demands are from CMP.

factor demands are from PMP : $\max_{K,L} \pi(K,L)$ // $pQ - TC$

e.g. $L^* = [p\alpha^{1-\beta}\beta^{\beta} w^{\beta-1} r^{-\beta}]^{\frac{1}{1-\alpha-\beta}}$

$\Rightarrow$ labor demand

L^*_{CMP}, K^*_{CMP} that minimizes cost don't need to maximize profit because cost is just one part of π

Duality : When optimization problems/solutions can be viewed from either of two perspectives :

Output Max Problem

$\max_{K,L} Q(K,L)$

s.t. $wL + rK = C = TC_{min}$

K^*_{OMP}, L^*_{OMP}, $Q^*_{max} = Q_0$

if $C = TC^*_{min}$ from CMP

then $Q_{max} = Q_0$

OMP

CMP

$\min_{K,L} TC(K,L)$

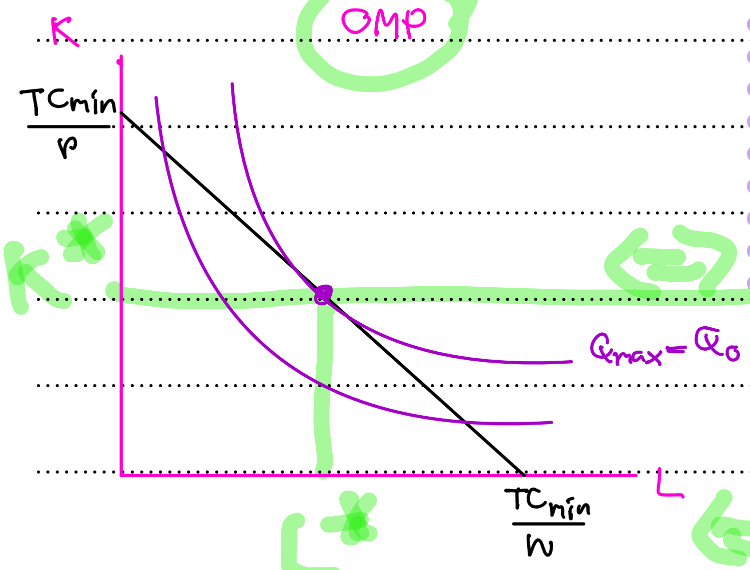
s.t. $Q(K,L) = Q_0 = Q^*_{max}$

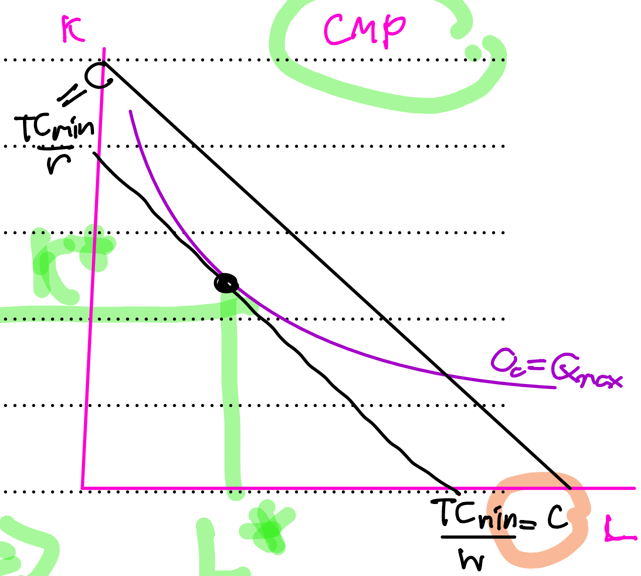
K^*_{CMP}, L^*_{CMP}, $TC^*_{min} = C$

if $Q_0 = Q^*_{max}$ from OMP

then $TC_{min} = C$

CMP





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Utility maximization subject to budget constraint

$$\max_{x,y} \quad U = U(x,y) \quad (U_x, U_y > 0)$$

$$\text{St.} \quad xP_x + yP_y = B \quad \text{"Budget Const."}$$

Step 1: Lagrangian Function

$$\max_{\lambda, x, y} \quad L = U(x,y) + \lambda\left[B - P_x x - P_y y\right]$$

Step 2: First-Order Condition (FONC):

$$\text{FOC:} \quad L_\lambda = \frac{\partial L}{\partial \lambda} = B - P_x x - P_y y = 0 \Rightarrow P_x x + P_y y = B \quad (1.)$$

$$L_x = \frac{\partial L}{\partial x} = \frac{\partial U}{\partial x} - \lambda P_x = 0 \Rightarrow MU_x = \lambda P_x \quad (2.)$$

$$L_y = \frac{\partial L}{\partial y} = \frac{\partial U}{\partial y} - \lambda P_y = 0 \Rightarrow MU_y = \lambda P_y \quad (3.)$$

(2.) / (3.)

$$MRS_{xy} = \frac{MU_x}{MU_y} = \frac{P_x}{P_y} \quad \text{—(4.)}$$

From (4.) & (1.), we will get $x^* = x(P_x, P_y, B)$

$$y^* = y(P_x, P_y, B)$$

"Marshallian / Walrasian / Uncompensated demand function"

Combines Income & substitution effects when price changes.

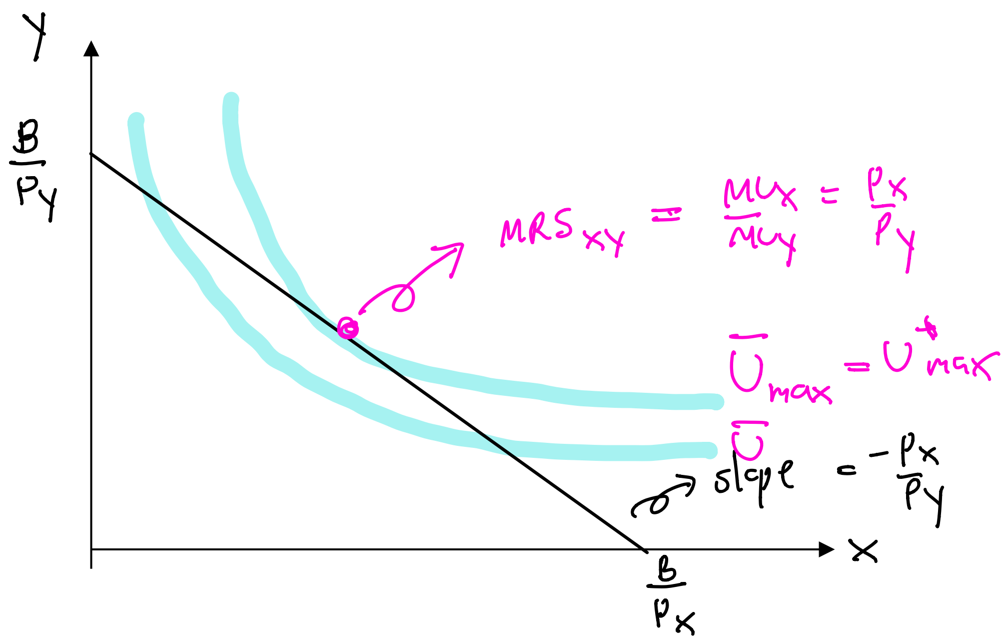
Economic Interpretation

Interpretation 1 : Marginal utility per one baht paid

$$\frac{MU_x}{P_x} = \frac{MU_y}{P_y} = \lambda^*$$

to max U s.t. B, consumer will chose so that MUs from each 1 baht spent on each good are equal.

Interpretation 2: Indifference Curve



Budget line : $P_x x + P_y Y = B \Rightarrow Y = \frac{B}{P_X} - \frac{P_x}{P_y} X$

IC : $U = U(x,y) = \bar{U}$

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = d\bar{U} = 0$$

$$\frac{dy}{dx} = -\frac{MU_x}{MU_y}$$

Meaning of Lagrange multiplier

$L^* = U(x^*, y^*) + \lambda^* [\underbrace{B - P_x x^* - P_y y^*}_{= 0 \Rightarrow L^* = U^*}]$

$\frac{\partial L^*}{\partial B} = \lambda^* = \frac{\partial U^*}{\partial B}$ = marginal utility of income (budget money B)

if B ↑ by 1 unit, maximized utility will increase by λ^* units.

Step 3 **Second-Order Sufficient Condition**

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} = & L_{\lambda x} = & L_{\lambda y} = \\ L_{x\lambda} = & L_{xx} = \frac{\partial^2 U}{\partial x^2} = & L_{xy} = \frac{\partial^2 U}{\partial x \partial y} = \\ L_{y\lambda} = & L_{yx} = \frac{\partial^2 U}{\partial y \partial x} = & L_{yy} = \frac{\partial^2 U}{\partial y^2} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n =__________ k = ___________

Check the last _________ leading principle Minor :

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Suppose that Mr. Musk's utility function takes the form:

$$U(x,y) = 10x^{\frac{1}{2}}y^{\frac{1}{2}}$$

Mr. Musk has m baht. Price of Good x is P_x, and price of good y is P_y. What will be the level of x and y that maximize the utility?

$$\max_{x,y} \quad 10x^{1/2}y^{1/2}$$

$$\text{s.t.} \quad P_x x + P_y y = m$$

Lagrangian function:

$$\max_{\lambda,x,y} \quad L = 10x^{1/2}y^{1/2} + \lambda[m - P_x x - P_y y]$$

FOC:

$$L_\lambda = \frac{\partial L}{\partial \lambda} = m - P_x x - P_y y = 0 \Rightarrow P_x x + P_y y = m \quad (1.)$$

$$L_x = \frac{\partial L}{\partial x} = 5x^{-1/2}y^{1/2} - \lambda P_x = 0 \Rightarrow 5x^{-1/2}y^{1/2} = \lambda P_x \quad (2.)$$

$$L_y = \frac{\partial L}{\partial y} = 5x^{1/2}y^{-1/2} - \lambda P_y = 0 \Rightarrow 5x^{1/2}y^{-1/2} = \lambda P_y \quad (3.)$$

(2.)/(3.)

$$\frac{x^{-1/2}y^{1/2}}{x^{1/2}y^{-1/2}} = \frac{P_x}{P_y}$$

$$\frac{y}{x} = \frac{P_x}{P_y} \quad —(4.)$$

$$y = \frac{P_x}{P_y}x \quad —(5.)$$

substitute (5.) into (1.)

$$P_x x + P_y\left(\frac{P_x x}{P_y}\right) = m$$

$$x^* = \frac{m}{2P_x}$$

$P_x \uparrow, \; x^* \downarrow$

$m \uparrow, \; x^* \uparrow$

SOC:

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Minimize Total Expenditure Under the Level of Utility

Suppose now that a consumer would like to have utility U_0. Her utility is $U = U(x, y)$.

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<u>Step 1</u> Lagrangian Function

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<u>Step 2</u> First-Order Condition (FONC):

<u>Step 3</u>: Second-Order Sufficient Condition

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n = ______ k = ______

Check the last________leading principle Minor:

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Income expansion path (Income consumption curve: ICC)

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Duality: Utility Maximization Problem vs. Expenditure Minimization Problem

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Other optimization problems:

Multiproduct firm with technological constraint

Quantities of two products are $x \& y$.

How many of $x \& y$ should the firm choose to produce to minimize the cost of production subjecting to technological constraint?

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Multiplant firm who knows exact total market demand

Let factory 1 produce Q_1 with cost $C_1 = C_1(Q_1)$; factory 2 produce Q_2 with cost $C_2 = C_2(Q_2)$; and let market demand be Q_0.

To minimize cost, how many should the firm produce at each factory given that it wants to meet the market demand?

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