Solution: Quiz 5

1. Find the solution set for
$$\frac{-x^2+x}{x-9} \le \left|\frac{3}{9-x} - \left|\frac{3}{x-9}\right|\right|.$$

Solution:

Notice that $\left|\frac{3}{x-9}\right| = \frac{3}{|x-9|}$ and $|x-9| = \begin{cases} -(x-9), & x-9<0 \Leftrightarrow x<9 \\ x-9, & x-9\ge 0 \Leftrightarrow x \ge 9 \end{cases}$, We will consider 2 cases for x: (I) $x \ge 9$ and (II) $x < 9$.

(I) For $x \ge 9$, $|x-9| = x-9$ and $|9-x| = -(9-x) = x-9$

$$\frac{-x^2+x}{x-9} \le \left|\frac{3}{9-x} - \left|\frac{3}{x-9}\right|\right| = \left|\frac{3}{9-x} - \frac{3}{x-9}\right| = \left|\frac{3}{9-x} + \frac{3}{9-x}\right| = \left|\frac{6}{9-x}\right| = \frac{6}{x-9}$$
$$\frac{-x^2+x}{x-9} \le \frac{6}{x-9}$$
$$\frac{-x^2+x-6}{x-9} \le 0 \quad \Rightarrow \quad \frac{x^2-x+6}{x-9} \ge 0$$

Notice that for $ax^2+bx+c = x^2-x+6$, since $b^2-4ac = 1-4(1)(6) < 0$, then $x^2-x+6>0$ for all real numbers x. That is, the inequality in this case becomes

$$\frac{1}{x-9} \ge 0 \quad \Rightarrow \quad x-9>0 \quad \Rightarrow \quad x>9$$

Therefore, the solution set in this case is $(9,\infty) \cap [9,\infty) = (9,\infty)$.

(II) For $x<9$, $|x-9| = -(x-9) = 9-x$ and $|9-x| = (9-x)$

$$\frac{-x^2+x}{x-9} \le \left|\frac{3}{9-x} - \left|\frac{3}{x-9}\right|\right| = \left|\frac{3}{9-x} - \frac{3}{9-x}\right| = 0$$
$$\frac{-x(x-1)}{x-9} \le 0 \quad \Rightarrow \quad \frac{x(x-1)}{x-9} \ge 0$$

we consider $x = 0, 1, 9$ to subdivide the intervals.

	$x \in (-\infty, 0)$	$x \in (0,1)$	$x \in (1,9)$	$(9,\infty)$
$\frac{(x)(x-1)}{x-9}$	$\frac{(-)(-)}{(-)} = \boxed{(-)}$	$\frac{(+)(-)}{(-)} = (+)$	$\frac{(+)(+)}{(-)} = \boxed{(-)}$	$\frac{(+)(+)}{(+)} = (+)$

That is, the solution set in this case is $\{[0,1] \cup (9,\infty)\} \cap (-\infty, 9) = [0,1]$.

From (I) and (II), the solution set is $(9,\infty) \cup [0,1] = [0,1] \cup (9,\infty)$.