B.E. International Program
Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics
Semester 1/2015

Homework 3 - Suggested Answers

Due 5 November 2015

Question 1

Calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ (if possible) of the following functions.

a. $f(x,y) = \frac{5xy^2}{x^2+y^2}$

b. $f(x,y) = \ln(x^2y + xy^2) - x^2 - y^2$

c. $f(x,y,z) = \mathrm{x}\mathrm{z}^2\ln(y) - \frac{y}{z^2+x-y}$

d. $f(x,y,z) = \mathrm{e}^{x+\ln(z)} - \ln(x^2)y^2z^3$

a. $\partial f/\partial x = \frac{(x^2+y^2)(5y^2)-(5xy^2)(2x)}{(x^2+y^2)^2}$

$\partial f/\partial y = \frac{(x^2+y^2)(10xy)-(5xy^2)(2y)}{(x^2+y^2)^2}$

b. $\partial f/\partial x = \frac{2xy+y^2}{x^2y+xy^2} - 2x$

$\partial f/\partial y = \frac{x^2y+2xy}{x^2y+xy^2} - 2y$

c. $\partial f/\partial x = z^2 ln(y) + [y/(z^2 + x - y)^2\ ]$
$\partial f/\partial y = xz^2/y - [(z^2 + x)/(z^2 + x - y)^2\ ]$
$\partial f/\partial z = 2xzln(y) + [2yz/(z^2 + x - y)^2\ ]$

d. $\partial f/\partial x = ze^x - (2/x)y^2z^3$
$\partial f/\partial y =- 2yln(x^2)z^3$
$\partial f/\partial z = e^x - 3z^2 ln(x^2)y^2$

Question 2

The optimal profit function of a firm can be given by,

$$\pi^*(p, w_1, w_2, A) = A * p^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}}$$

where $0 < \beta < 1$ and $\gamma < 1$, A is the level of technology, P is price of output, w_1 is the factor price of capital, and w_2 is the factor price of labor.
Consider the following problem

a. Use the partial derivative to conclude about the relationship between price and the level of profit.

$\partial\pi^*/\partial p = \frac{1}{1-\beta}A * p^{\frac{\beta}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}} > 0$

b. How does the technical progress affect the level of profit? Show your result by using the partial derivative.

$\partial\pi^*/\partial A = p^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}} > 0$

c. How does the level of factor price of inputs affect the level of profit? Show your result for both types of input, using the partial derivative. Then, explain the intuition of your result in economics.

$$\partial\pi^*/\partial w_1 = \left[\frac{\beta}{(\beta-1)}\right] Ap^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}-1} * w_1{}^{\gamma-1} < 0$$

$$\partial\pi^*/\partial w_2 = \left[\frac{\beta}{(\beta-1)}\right] Ap^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}-1} * w_2{}^{\gamma-1} < 0$$

This is because $0 < \beta < 1 \Rightarrow \beta - 1 < 0 \Rightarrow \left[\frac{\beta}{(\beta-1)}\right] < 0$

d. Show that the profit function is convex in factor price of inputs. That is, the second-order partial derivatives of profit with respect to factor price of both capital and labor are greater than zero.

$$\partial^2\pi^*/\partial w_1{}^2 = \left[\frac{\beta}{(\beta-1)}\right](\gamma - 1)\, Ap^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}-1} * w_1{}^{\gamma-2}$$
$$+ \left[\frac{\beta}{(\beta-1)}\right]\left[\frac{\beta}{\gamma(\beta-1)} - 1\right] Ap^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}-2} * \gamma w_1{}^{2(\gamma-1)} > 0$$

$$\partial^2\pi^*/\partial w_2{}^2 = \left[\frac{\beta}{(\beta-1)}\right](\gamma - 1)\, Ap^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}-1} * w_2{}^{\gamma-2}$$
$$+ \left[\frac{\beta}{(\beta-1)}\right]\left[\frac{\beta}{\gamma(\beta-1)} - 1\right] Ap^{\frac{1}{1-\beta}}(w_1{}^{\gamma} + w_2{}^{\gamma})^{\frac{\beta}{\gamma(\beta-1)}-2} * \gamma w_2{}^{2(\gamma-1)} > 0$$

Note that $\left[\frac{\beta}{(\beta-1)}\right] < 0$, $(\gamma - 1) < 0$ and $\left[\frac{\beta}{\gamma(\beta-1)} - 1\right] < 0$.

Remarks: The was a typo in the question, regarding to the assumption for "γ". The question was supposed to be written by assuming that $\gamma < 1$. In text, I write $\gamma > 1$, which makes it impossible to show the convexity of the profit function.

Question 3

Consider a simple macroeconomic model given below,

$$Y = C + G$$
$$C = C_0 + c_1(Y - T)$$
$$T = T_0 + t_1 Y$$
$$G = G_0$$

where Y is national income, C is consumption, T is the amount of tax collected, G is the level of government expenditure.

Answer the following questions:

a. State all the endogenous variables. What are the parameters and exogenous variables in the model?

   Endogenous: C, Y, T

   Exo and Parameters: C_0, c_1, T_0, t_1, G_0

b. Derive the equilibrium solution of all the endogenous variables?

   $$Y^* = \frac{C_0 - c_1 T_0 + G_0}{1-(1-t_1)c_1}$$
   $$T^* = T_0 + t_1 \left[\frac{C_0 - c_1 T_0 + G_0}{1-(1-t_1)c_1}\right]$$
   $$C^* = C_0 + c_1 \left[(1 - t_1)\frac{C_0 - c_1 T_0 + G_0}{1-(1-t_1)c_1} - T_0\right]$$

c. Use the partial derivative to show the effect change in c_1 and G_0 on the equilibrium of endogenous variables?

$$\partial Y^*/\partial c_1 = \frac{[1-(1-t_1)c_1](-T_0)+(C_0-c_1T_0+G_0)(1-t_1)}{(1-(1-t_1)c_1)^2} = \frac{(C_0+G_0)(1-t_1)-T_0}{(1-(1-t_1)c_1)^2} > 0$$
$$\partial Y^*/\partial G_0 = \frac{1}{1-(1-t_1)c_1} > 0$$

$$\partial T^*/\partial c_1 = t_1 \frac{(C_0+G_0)(1-t_1)-T_0}{(1-(1-t_1)c_1)^2} > 0$$
$$\partial T^*/\partial G_0 = \frac{t_1}{1-(1-t_1)c_1} > 0$$

$$\partial C^*/\partial c_1 = (1-t_1)\left[\frac{C_0-c_1T_0+G_0}{1-(1-t_1)c_1} + c_1\frac{(C_0+G_0)(1-t_1)-T_0}{(1-(1-t_1)c_1)^2}\right] - T_0 > 0$$

$$\partial C^*/\partial G_0 = \frac{c_1(1-t_1)}{1-(1-t_1)c_1} > 0$$

d. How does the marginal propensity to tax (t_1) affect the equilibrium level of income (Y*) and consumption (C*)?

$$\partial Y^*/\partial t_1 = -\frac{c_1}{(1-(1-t_1)c_1)^2} < 0$$

$$\partial C^*/\partial t_1 = -c_1\left[\frac{C_0-c_1T_0+G_0}{1-(1-t_1)c_1} + (1-t_1)\frac{c_1}{(1-(1-t_1)c_1)^2}\right] < 0$$

Question 4

Write the Hessian Matrix for each of the following functions:

a. $U(x,y) = 7x^2 + 8xy + 3y^2$

Ans. $H = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 8 & 6 \end{bmatrix}$

b. $z(x,y) = 5(13x - 5y)^2$

Ans. $H = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix} = \begin{bmatrix} 1690 & -650 \\ -650 & 250 \end{bmatrix}$

c. $f(x,y) = 4x^3 - 11xy - 7y^5$

Ans. $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 24x & -11 \\ -11 & -140y^3 \end{bmatrix}$

d. $Q(K,L) = (2K+1)(3L^2+2)$

Ans. $H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix} = \begin{bmatrix} 0 & 12L \\ 12L & 6(2K+1) \end{bmatrix}$

Question 5

The demand for a product depends on the price p_1 of the product and on the price p_2 charged by a competing producer, and it is given by:

$$D(p_1, p_2) = 36 - \frac{8p_1}{\sqrt{p_2}} .$$

a. Find $\frac{\partial D}{\partial p_1}$ and $\frac{\partial D}{\partial p_2}$, and comment on the signs of the partial derivatives.

Ans. $\frac{\partial D}{\partial p_1} = -8(p_2)^{-1/2} < 0$, showing that the demand decreases as its own price increases.

$\frac{\partial D}{\partial p_2} = -(8)\left(-\frac{1}{2}\right)p_1(p_2)^{-\frac{3}{2}} = 4p_1(p_2)^{-\frac{3}{2}} > 0$, showing that the demand increases as the price of a competing product increases.

b. Calculate the own-price and cross-price elasticities of demand when $p_1 = 3$ and $p_2 = 4$.

Ans. When $p_1 = 3$ and $p_2 = 4$, $D = 36 - (8)(3)/2 = 24$

$$\varepsilon_{D,p_1} = \frac{\partial D/D}{\partial p_1/p_1} = \frac{\partial D}{\partial p_1} \cdot \frac{p_1}{D} =- \frac{8}{2} \cdot \frac{3}{24} =- 0.5$$

Question 6

Suppose the production function Q depends on the number of workers L according to the formula:

$$Q = Lg\left(\frac{ln(L)}{L}\right)$$

where g(.) is a differentiable function. Find expressions for $\frac{dQ}{dL}$ and $\frac{d^2Q}{dL^2}$.

Ans. $\frac{dQ}{dL} = g'(u)\left[\frac{1}{L} - \frac{\ln(L)}{L}\right] + g(u)$ where $u = \frac{\ln(L)}{L}$

$$\frac{d^2Q}{dL^2} = g'(u)\left[-\frac{1}{L^2} - \left(\frac{1}{L} - \frac{\ln(L)}{L}\right)\right] + \left[\frac{1}{L} - \frac{\ln(L)}{L}\right]g''(u)\frac{du}{dL} + g'(u)\frac{du}{dL} = -\frac{g'(u)}{L^2} + \left[\frac{1}{L} - \frac{\ln(L)}{L}\right]^2 g''(u)$$