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4 Testing Hypotheses about a Single Linear Combination of the Parameter

Consider

$$\log(wage) = \beta_0 + \beta_1 jc + \beta_2 univ + \beta_3 \exp er + u$$

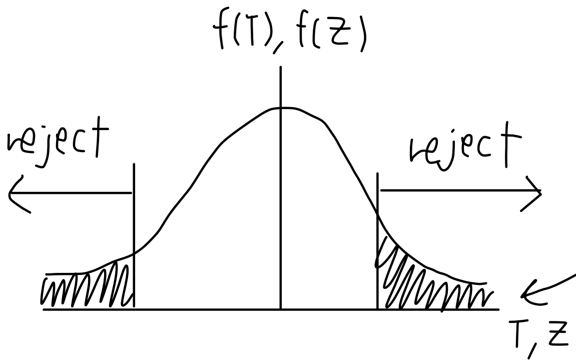
where $jc =$ number of years attending a two-year college
$univ$ = number of years at a four-year college
$\exp er$ = months in the workforce.
We want to test whether $\beta_1 = \beta_2$. → if the return from 1 more year of education at junior college is the same as that of university

$H_0 : \beta_1 = \beta_2 \rightarrow H_0 : \beta_1 - \beta_2 = 0$

against

$H_a : \beta_1 \neq \beta_2 \rightarrow H_a : \beta_1 - \beta_2 \neq 0$

2 – tailed test



area = $\frac{\text{significant level}}{2}$

$$t = \frac{(\hat{\beta}_2 - \hat{\beta}_1) - 0}{2}$$

we compute this t-statistics and compare with the critical value

where $s.e(\hat{\beta}_2 - \hat{\beta}_1) = \sqrt{\widehat{var}(\hat{\beta}_2 - \hat{\beta}_1)}$

$$= \sqrt{\widehat{var}(\hat{\beta}_1) + \widehat{var}(\hat{\beta}_2) - 2\,\widehat{cov}(\hat{\beta}_1, \hat{\beta}_2)}$$

Not very straight forward to calculate
↓
we use variable transformation tricks
↓
see notes

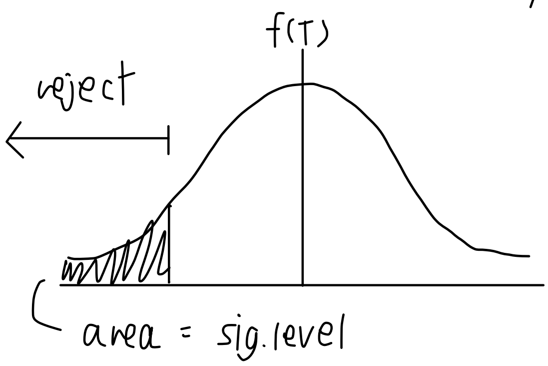
another possible hypothesis test (one-tailed alternative)

$H_0: \beta_1 = \beta_2 \rightarrow H_0: \beta_1 - \beta_2 = 0$

$H_a: \beta_1 = \beta_2 \rightarrow H_a: \beta_1 - \beta_2 < 0$

It would assume that β_1 would not be more than β_2

(Returns to a 2-year college would never be more than return to university education)



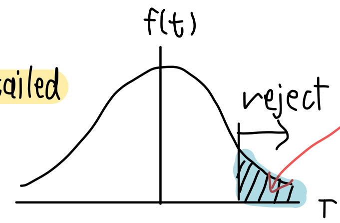
$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{s.e.(\hat{\beta}_1 - \hat{\beta}_2)}$$

Then, go to the extra note

5 Computing p-Values for t-Tests

- What is the significance level given the computed t-statistics?

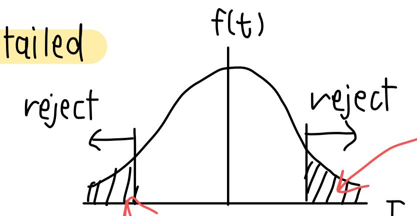
1-tailed



This shaded area in the rejection region is the p-value

$$t = \frac{\hat{\beta}_j - \beta_j}{s.e.\ \hat{\beta}_j}$$

2-tailed



Total area from 2 sides

$$t = \text{computed } t = \frac{\hat{\beta}_j - \beta_j}{s.e.\ \hat{\beta}_j}$$

- p-value : $P(|T| > |t|)$

T = t-distributed random variable with d.f. $= n-k-1$

t = computed t-statistics

→ p-value = probability that a random T value will be greater (in the || term) than our t in the H_0 test

In-class exercise

consider the multiple regression model, assume MLR 1-6 are satisfied

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + u$$

You would like to test the $H_0: \beta_1 - 3\beta_2 = 1$

H_a: otherwise is true

1st write t-statistics for testing H_0

$$t = \frac{(\hat\beta_1 - 3\hat\beta_2) - 0}{s.e.(\hat\beta_1 - 3\hat\beta_2)}$$

2nd Define $\theta_1 = \hat\beta_1 - 3\hat\beta_2$ → $H_0: \theta_1 = 1$, $H_a: \theta_1 \neq 1$

$$t = \frac{\hat\theta_1 - 1}{s.e.(\hat\theta_1)}$$

→ we need our regression to have θ_1 in it.
so, STATA or OLS estimation will automatically give $\hat\theta_1$ & s.e. $\hat\theta_1$

Now, $\hat\beta_1 = \hat\theta_1 + 3\hat\beta_2$ or $\beta_1 = \theta_1 + 3\hat\beta_2$

Substitute in the main regression and get

$$y = \beta_0 + (\theta_1 + 3\beta_2) X_1 + \beta_2 X_2 + \beta_3 X_3 + u$$

$$= \beta_0 + \theta_1 X_1 + 3\beta_2 X_1 + \beta_2 X_2 + \beta_3 X_3 + u$$

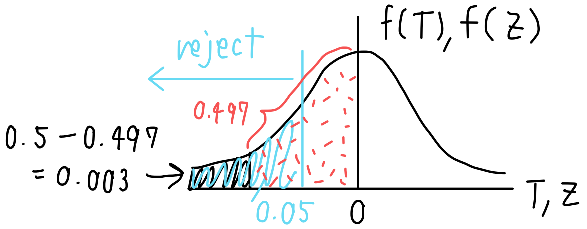
$$= \beta_0 + \theta_1 X_1 + \beta_2 (X_2 + 3X_1) + \beta_3 X_3 + u$$

⭐ Now, the explanatory variables are going to be

X_1, $X_2 + 3X_1$, and X_3

- we can calculate $t = \dfrac{\hat{\theta}_1 - 1}{s.e.\ \hat{\theta}_1}$

Example 1: $H_0 : \beta_j \geq 0$, $H_a : \beta_j < 0$, d.f.= 140. → z - table



→ p-value = what should be the significant level, given the critical value of − 2.75 ??

→ Find the shaded area

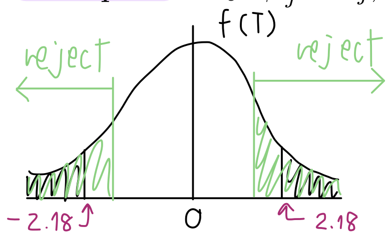
suppose the calculated $t_{\hat{\beta}_j} = -2.75$

$= (\hat{\beta}_j - \beta_j) / s.e.(\hat{\beta}_j)$

- From the z-table, the value -2.75 corresponds to area = 0.003.
- Thus, p-value = 0.003.
- Would we reject H_0 if we use the significance level = 5%?

★ RULE !: we reject Ho if p-value < sig. level ★

Example 2: $H_0 : \beta_j = a_j$, $H_a : \beta_j \neq a_j$, $\underbrace{\text{d.f.} = 18.}_{\text{use t-table}}$



suppose the calculated $t_{\hat{\beta}_j} = -2.18$

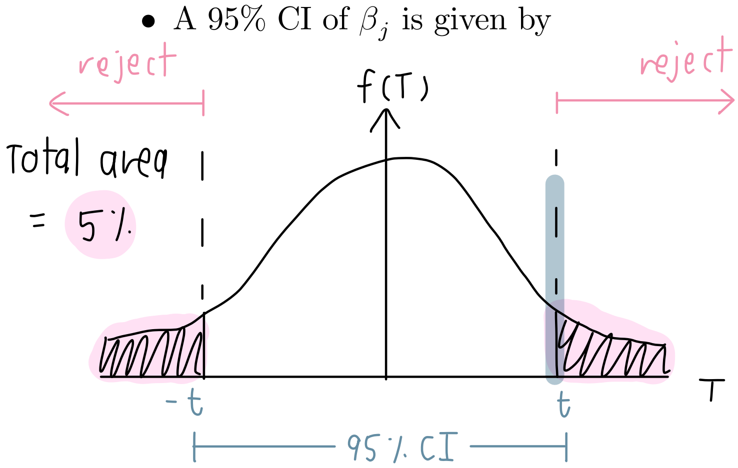
- From the t-table, the value -2.18 corresponds to area = 0.02 to 0.05.
- Thus, p-value = is between 0.02 − 0.05
- Would we reject H_0 if we use the significance level = 5%?

6 Confidence Intervals (CI)

- Confidence Interval s for the POPULATION PARAMETER (β_j)

↳ The range of value that would capture the true β_j at a 95% chance

- A 95% CI of β_j is given by



CI → $\hat{\beta}_j \pm c \times s.e.(\hat{\beta}_j)$

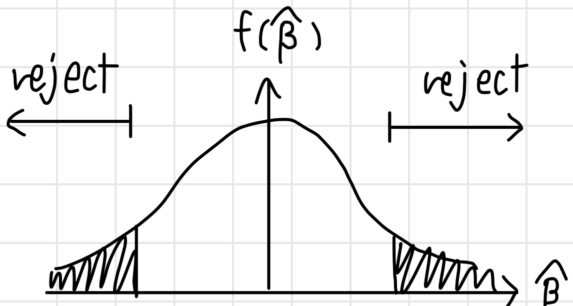
c is the $\boxed{97.5}$ percentile in the t- distribution with n−k−1 d.f.

Inference → Hypothesis testing about "β" the true parameter

$$wage = \beta_0 + \beta_1 educ + \beta_2 experience + \dots + w$$

we want to test hypotheses about the true impact (β) of each x variables (educ, experience) on the dependent variables (y)

BUT we don't know what the true β are. So, we use $\hat{\beta}$ (estimator) and s.e.($\hat{\beta}$) to test the hypotheses



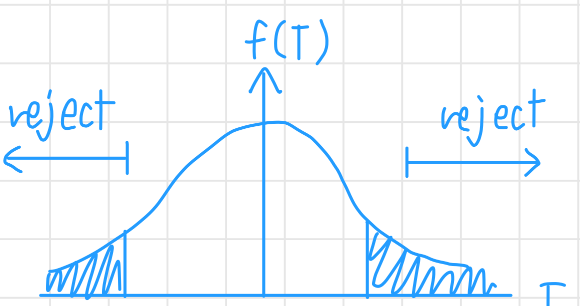
β = a hypothesized value

ex. $\beta = 0$ or $\beta = 1$ etc.

1 Test if β = same number

eg. $\beta_j = 0$ → X_j has no impact on y

$\beta_j = 1$ → 1 unit ↑ in X_j correspond to 1 unit ↑ in y

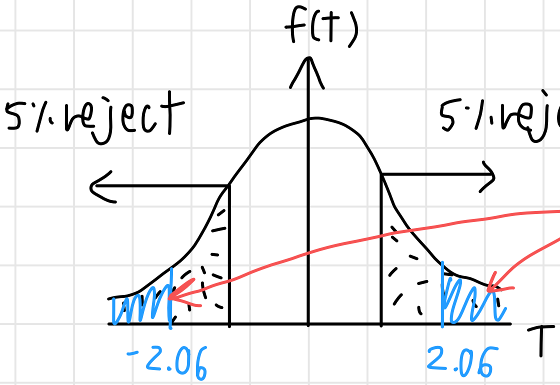


$$0 = \frac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)}$$

⇒ t-test ⭐ → How ??

$$\frac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)} \sim t_{d.f.}$$

$\boxed{\text{Significant Level = total area in rejection region}}$



ass. df. = 100

area $= 2(0.5 - 0.4803)$
$= 2 \times 0.0197$
$= 0.0394$ = p-value!

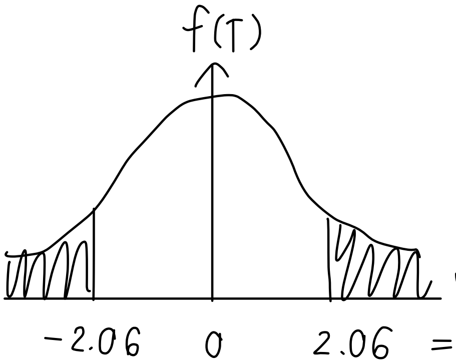
- Suppose, we calculate a t-statistic $= \dfrac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)} = 2.06$
- Suppose, we are testing $\underbrace{H_0: \beta_j = 0, \quad H_a: \beta_j \neq 0}_{\text{2-tailed test}}$
- p-value = total shaded area

p-value = significant level which we will reject the Ho or prob. that we will reject Ho

* If p-value < significance level → reject Ho

Example 1: **95% CI** d.f. = 25

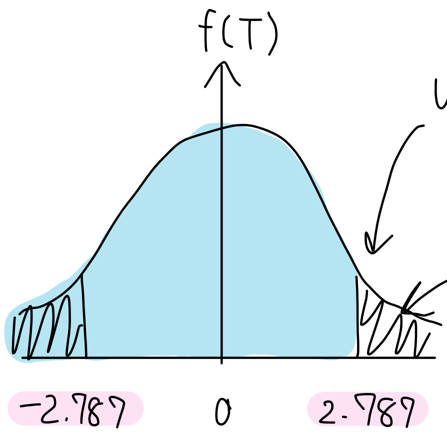
*ตอนสอบอาจมีหา s.e. (เอง)



= value of 97.5^{th} percentile in the t_{25} distribution

The 95% CI for $\hat{\beta}_j$ = $[\hat{\beta}_j - 2.06 \cdot s.e.(\hat{\beta}_j), \hat{\beta}_j + 2.06 \cdot s.e.(\hat{\beta}_j)]$

Example 2: **99% CI** d.f. = 25



what pctile?? 99.5

or area = 1−0.005 = 0.995

area = $\frac{1\%}{2} = \frac{0.01}{2} = 0.005$

The 99% CI for $\hat{\beta}_j$ = $[\hat{\beta}_j - 2.787 \cdot s.e.(\hat{\beta}_j), \hat{\beta}_j + 2.787 \cdot s.e.(\hat{\beta}_j)]$

12 Mar 2020

19 Mar 2020

F-test motivation

→ we want to test the significance of a group of hypotheses (multiple hypotheses)

$$Grade_{325} = \beta_0 + \beta_1 \#times\_front + \beta_2 \#times\_back$$
$$+ \beta_3 hr\_study + \beta_4 past\_GPA + \beta_5 gender + u$$

Ho : seat position doesn't have impact on GPA

$\beta_1 = 0$ and $\beta_2 = 0$ ⇒ $\beta_1 = \beta_2 = 0$

testing 2 βs at the same time → F test

Ha : seat position matters

$$\left.\begin{array}{l} \beta_1 \neq 0 \text{ and } \beta_2 \neq 0 \\ \text{or } \beta_1 \neq 0 \text{ and } \beta_2 = 0 \\ \text{or } \beta_1 = 0 \text{ and } \beta_2 \neq 0 \end{array}\right\} \text{at least one of the } \beta_1, \beta_2 \neq 0$$

19 Mar 2020

7 Testing Multiple Linear Restrictions: The F-test

Suppose the model is specified by

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$
$$H_0 : \beta_1 = 0 \text{ and } \beta_2 = 0$$
$$Ha,\ H_1 : H_0 \text{ is not true}$$

→ want to test if X_1 and X_2 BOTH have no impact on Y

We can use the F-test to test this type of "multiple hypotheses".

1. Our full model is called the "unrestricted" model (ur). Suppose it can be expressed as:

   ↙ big model

   $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + u$ is true ⇒ Reject Ho

   $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + ... + \beta_k x_k + u$$

2. The model which takes out x (which we think its associated $\beta = 0$) is called the restricted model (r). ← small model

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + u$ is true ⇒ Reject Ho

- suppose there are "q" number of β that we would like to perform a joint-test of = 0 e.g. in this model q = 2

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_{K-q} X_{K-q} + u$$

H_0 : $\beta_{K-q+1} = \beta_{K-q+2} = \beta_K = 0$

H_a : H_0 is not true

$$\underbrace{\underbrace{y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_{K-q} X_{K-q}}_{r} + \beta_{K-q+1} X_{K-q+1} + \beta_{K-q+2} X_{K-q+2} + \beta_K X_K + u}_{ur}$$

$$F \equiv \frac{\frac{(SSR_r - SSR_{ur})}{q}}{\frac{SSR_{ur}}{(n-K-1)}}$$

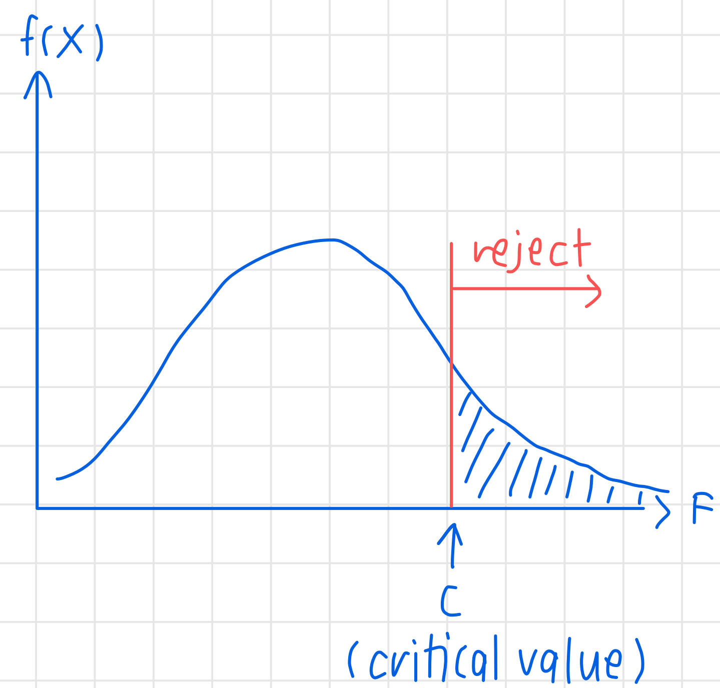
← This whole term is always ⊕ because $SSR_{ur} < SSR_r$. Every time you add 1 more x, the model will be better explained

$(n-K-1)$ ← d.f. of the ur. model

so, if everytime you add 1 more X variable, the SSR $\downarrow$ and $R^2 \uparrow$, why don't we just keep an additional X in the model

$\rightarrow$ because everytime we add 1 more X, $Var(\hat{\beta}_3)$ will increase, making the prediction of β less precise. so, we only keep the additional X, if it/they can improve the model enough

- can reduce SSR ($\uparrow R^2$) enough
- can significantly reduce SSR and $\uparrow R^2$



$H_0: \beta_2 = \beta_3 = \ldots = 0$

H_a. H_0 not true

$F \sim F_{q, n-k-1}$

# of joint hypotheses being tested

d.f. of the ur model

⭐ we reject H_0 of jointly no effect if $F > c$

3. Some useful facts

1 $R^2_{ur} > R^2_r$ because any additional X would increase R^2 (improve fit) $\rightarrow SSR_{ur} < SSR_r$

2 By including more X, the model is certainly better explained. However, we would like to reject Ho if the inclusion of extra variables does not improve the model enough

4. Other ways to calculate the F-statistics:

$\Rightarrow$ From $R^2 = 1 - \frac{SSR}{SST}$ SSR $\rightarrow$ RSS, SST $\rightarrow$ TSS

we have
$$F \equiv \frac{(R^2_{ur} - R^2_r)}{q} \Big/ \frac{(1 - R^2_{ur})}{n-k-1}$$

of β that are set to "0" (q)

of obs. (n)

# of slope β (k)

intercept (1)

$\rightarrow$ If we want to test the overall significance of the model
$H_0: \beta_1 = \beta_2 = \beta_3 = \ldots = \beta_K = 0$, H_a: otherwise

$$F \equiv \frac{R^2/K}{(1-R^2)/(n-k-1)}$$

R^2 of the model $\approx$ ur
the "r" model has no X at all

Example: Suppose we are interested in understanding the determinant of a baseball player's salary.

	salary	= season salary
ur	*years*	= years in major leagues
	gamesyr	= games per year in the league
u	*bavg*	= career batting average
	hrunsyr	= homeruns per year
	rbisyr	= runs batted in per year

If we want to test whether performance has any impact on salary

$H_0: \beta_{bavg} = \beta_{hrunsyr} = \beta_{rbisyr} = 0$

H_a: otherwise is true

- the unrestricted model (ur) is defined by

The more explanatory variables = The more R^2

ur model

y | X (more set of explanatory variables)

```
. regress log_salary years gamesyr bavg hrunsyr rbisyr
```

	Source	SS	df	MS
SSE	Model	308.989208	5	61.7978416
SSR	Residual	183.186327	347	.527914487
SST	Total	492.175535	352	1.39822595

Number of obs	= 353
F(5, 347)	= 117.06
Prob > F	= 0.0000
R-squared	= 0.6278
Adj R-squared	= 0.6224
Root MSE	= .72658

	log_salary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
1	years	.0688626	.0121145	5.68	0.000	.0450355	.0926898
2	gamesyr	.0125521	.0026468	4.74	0.000	.0073464	.0177578
3	bavg	.0009786	.0011035	0.89	0.376	-.0011918	.003149
4	hrunsyr	.0144295	.016057	0.90	0.369	-.0171518	.0460107
5	rbisyr	.0107657	.007175	1.50	0.134	-.0033462	.0248776
intercept →	_cons	11.19242	.2888229	38.75	0.000	10.62435	11.76048

when considering each of the performance X one-by-one, none of them has a significant impact at 5%.

- the restricted model (r) is defined by

y | X

```
. regress log_salary years gamesyr
```

Source	SS	df	MS
Model	293.864058	2	146.932029
Residual	198.311477	350	.566604221
Total	492.175535	352	1.39822595

Number of obs	= 353
F(2, 350)	= 259.32
Prob > F	= 0.0000
R-squared	= 0.5971
Adj R-squared	= 0.5948
Root MSE	= .75273

log_salary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
years	.071318	.012505	5.70	0.000	.0467236	.0959124
gamesyr	.0201745	.0013429	15.02	0.000	.0175334	.0228156
_cons	11.2238	.108312	103.62	0.000	11.01078	11.43683

Now, our H_0 and H_a becomes

- But when performing an F-test, performances have joint impact

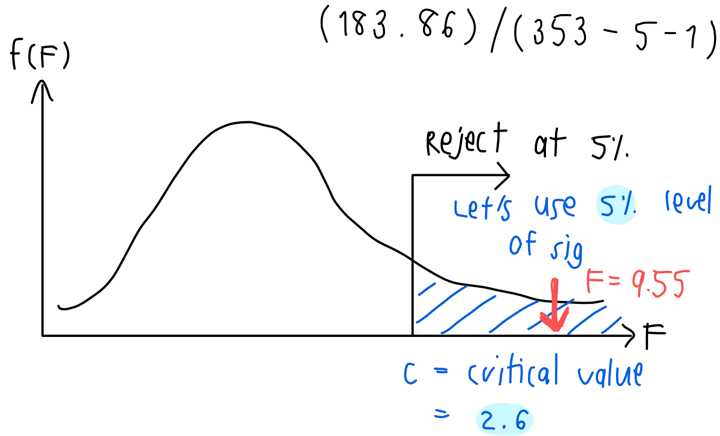
$$F \equiv \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}$$

$$= \frac{(198.311 - 183.186)/3}{(183.86)/(353-5-1)} \approx 9.55$$

HW:

$$F \equiv \frac{(R^2/q)}{(1-R^2)/n-k-1}$$

$$\equiv ???$$



Since F = 9.55 > 2.6, we reject H_0 at 5% level and conclude that performances have joint effects on salary

8 How the Hypothesis Testing is done in Practice

1. Check the values of $t - statistic$ reported by the statistical software (i.e. STATA, SPSS, SAS)

   $\Rightarrow$ These $t - statistics$ are to test $H_0 : \beta_i = 0$ z table

   $\Rightarrow$ If the d.f. > 30, then when $t \;\; > 1.96$, we can reject H_0 with 5% sig. level

   $\Rightarrow$ **When** $t \;\; >$ **1.96**, we can say that β_i is **statistically significant** at 5% level. (value of $\beta_i \neq 0$)

   $\Rightarrow$ **When** $t \;\; <$ **1.96** we can say that β_i is **not statistically significant** at 5% level.

   $\Rightarrow$ If $t \;\; < 1.96$ we can drop x_i from the model

   $\Rightarrow$ After we drop x_i, we estimate the new regression function and obtain a new set of $\hat{\beta}$.

2. We can also perform other hypothesis testings of interest.

   e.g. $H_0 : \beta_i = \beta_j$

   or $\quad H_0 : \beta_i = 5 \quad$ etc.

   or perform an F-test for testing multiple linear restrictions

3. Usually, in economics, the estimation results are reported using this form

more explanatory variables

sales →

company performance

CEO characteristics

Dependent Variable: log(*salary*)			
Independent Variables	**(1)**	**(2)**	**(3)**
log(*sales*)	.224 (.027)	.158 (.040)	.188 (.040)
log(*mktval*)	——	.112 (.050)	.100 (.049)
profmarg	——	−.0023 (.0022)	−.0022 (.0021)
ceoten	——	——	.0171 (.0055)
comten	——	——	−.0092 (.0033)
intercept	4.94 (0.20)	4.62 0.25)	4.57 (0.25)
Observations *R*-squared	177 .281	177 .304	177 .353

like a simple regression with 1 X

Multiple Regression Analysis : Further Issues

1 Data scaling on OLS statistics

Different unit of measurement

When we change the unit of measurment of a variable, the value of estimators would chance accordingly. For example

$$\widehat{bweght} = \widehat{\beta}_0 + \widehat{\beta}_1 cigs + \widehat{\beta}_2 fa\min c,$$

where
$bwght$ = child birth weight, in grams.
$cigs$ = number of cigarettes smoked by the mother while pregnant, per day.
$fa\min c$ = annual family income, in thousands of dollars.

- what if we use b weight in kilogram?

$1\ kg = 1{,}000\ g$

$$\widehat{bweight}_{kg} = \frac{\widehat{bweight}_g}{1000} = \frac{\hat\beta_0}{1000} + \frac{\hat\beta_1}{1000} cigs + \frac{\hat\beta_2}{1000} faminc$$

$$= \hat\alpha_0 + \hat\alpha_1 cigs + \hat\alpha_2 faminc$$

$$\rightarrow \hat\alpha_0 = \frac{\hat\beta_0}{1000}, \quad \hat\alpha_1 = \frac{\hat\beta_1}{1000}, \quad \hat\alpha_2 = \frac{\hat\beta_2}{1000}$$

- what if we use faminc in USD (instead of 1000 USD)

$$bweight_g = \hat\beta_0 + \hat\beta_1 cigs + \frac{\hat\beta_2}{1000} faminc_{USD}$$

The value of this variable is going to be 1000 time larger than faminc

$$= \hat\beta_0 + \hat\beta_1 cigs + \hat\theta_2 faminc_{USD}$$

$$\rightarrow \hat\theta_2 = \frac{\hat\beta_2}{1000}$$

in other words $\hat\theta_2$ = impact of 1 USD↑

$\hat\beta_2$ = " —— " 1,000 USD↑ in income

- what if we use bweght in kg and income in THB

$$bwght_{kg} = \frac{\hat{\beta}0}{1000} + \frac{\hat{\beta}1}{1000} cigs + \underbrace{\left(\frac{\hat{\beta}2}{1000}\right)}_{30{,}000} faminc_{THB}$$

This value is going to be 30,000 times more than faminc

$$\frac{d\ln x}{dx} = \frac{1}{x}$$

$$\downarrow$$

$$d\ln(x) = \frac{1}{x}dx$$

2 More on functional forms

- Logarithmic Functional Form

usually means natural log

$$\log(y) = \beta_0 + \beta_1 \log(x_1) + \beta_2 x_2 + u$$

$\Delta y = y_1 - y_2$

$\Delta x = x_{11} - x_{12}$

$$\circ\ \beta_1 = \frac{d\log(y)}{d\log(x_1)} = \frac{\frac{1}{y}dy}{\frac{1}{x_1}dx_1} = \frac{\frac{1}{y}\Delta y}{\frac{1}{x_1}\Delta x_1} = \frac{100 \times \frac{1}{y}\Delta y}{100 \times \frac{1}{x_1}\Delta x_1} = \frac{\%\ y}{\%\ x}$$

with the log y and log x format, the coefficient is going to be the elasticity ! (x elasticity of y)

price demand

$$\circ\ \beta_2 = \frac{d\log(y)}{dx_2} = \frac{\frac{1}{y}dy}{dx_2} = \frac{\frac{1}{y}\Delta y}{\Delta x_2}$$

→ If we want the upper term to be % change,

then $100\beta_2 = \frac{100\frac{1}{y}\Delta y}{\Delta x_2}$

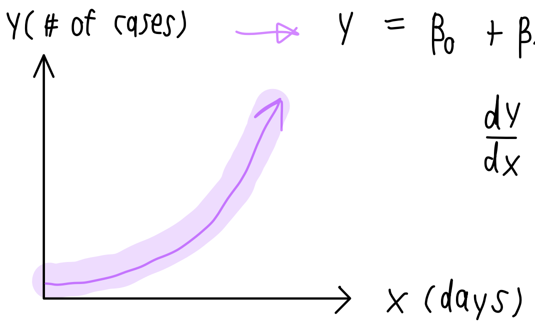
$100\beta_2 = \frac{\%\Delta y}{\Delta x_2}$

$100\beta_2$ = % Δ in y given that x_2 increases by 1 unit

- Models with Quadratics (squares)

→ capture increasing / decreasing marginal effects (slope of the relationship between x and y is not constant)

COVID-19 example

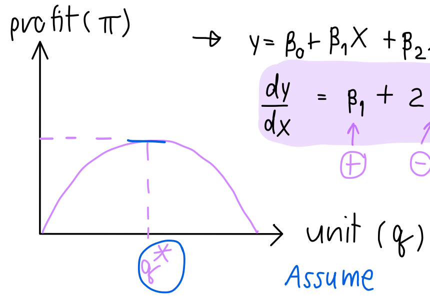


$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$$

$$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$$

⊕ ⊕ days

Decreasing returns



→ $y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$

$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$

⊕ ⊖

Assume

$\pi = (p - mc)q$; $mc = 10$

demand : $p = 100 - q$

F.O.C $= \frac{\partial \pi}{\partial q} = 0 = 90 - 2Q$

β_1 is ⊕ β_2 is ⊖

Example : Effects of Pullution on Housing Prices

$$\log(price) = \beta_0 + \beta_1 \log(nox) + \beta_2 \log(dist) + \beta_3 rooms + \beta_4 room^2 + \beta_5 stratio + u$$

where
price = housing price
nox = level of pollution
dist = distance from downtown
rooms = number of rooms
stratio = average student per teacher ratio
The estimation result is given by

In the U.S. or many other countries, student can apply to schools in the area without having to test any test.
So, the lower # student ratio, the better the school

```
regress  lprice lnox dist rooms rooms_sq stratio
```

Source	SS	df	MS
Model	51.4933152	5	10.298663
Residual	33.0889098	500	.06617782
Total	84.582225	505	.167489554

Number of obs =	506
F(5, 500) =	155.62
Prob > F =	0.0000
R-squared =	0.6088
Adj R-squared =	0.6049
Root MSE =	.25725

log (price)
log (nox)

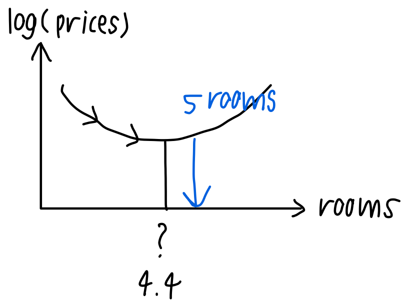
lprice		Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
lnox	β_1	-.9767545	.0995938	-9.81	0.000	-1.172429	-.7810806
dist	β_2	-.0321972	.0094013	-3.42	0.001	-.050668	-.0137264
rooms	β_3	-.5528032	.1612965	-3.43	0.001	-.8697056	-.2359007
rooms_sq	β_4	.0624697	.0124867	5.00	0.000	.0379368	.0870025
stratio	β_5	-.0486667	.0058131	-8.37	0.000	-.0600879	-.0372455
_cons		13.59154	.5650901	24.05	0.000	12.4813	14.70178

Non of them have | | less than 1.96, so each of them is statistically significant $|t| > 1.96$

All of them are less than 0.05, so we can reject the null of the coeff = 0 with 5% confident level all < 0.05

Consider the effect of "room"

$$\frac{d \log(price)}{d rooms} = \beta_3 + 2\beta_4 rooms = -0.553 + 2(0.062) \cdot rooms$$



at how many rooms does 1 additional room has a positive impact on log(price)??

$0 = -0.553 + 2(0.062) \cdot rooms$

$rooms = 4.4$

Answer → at 4.4 rooms or more

round up → at 5 rooms or more

What wold be the % change in price when the number of room increases from 5 to 6?

$$\frac{d \log(price)}{d rooms} = -0.553 + 2(0.062) rooms$$

$$\frac{100 \cdot \frac{1}{price} dprice}{drooms} = 100(-0.553 + 2(0.062) \cdot 5)$$

$= 100 \times 0.067 = 6.7\%$ increase.

what about % in price when #rooms increases from 5 to 7 ??

$$\% \Delta \text{ in price} = 100(-0.553 + 2(0.062)\cdot 6)$$
$$= 19.1\%$$

is total $\% \Delta$ in price when #rooms $\uparrow$ from 5 to 7 is $6.7 + 19.1\% = 25.8\%$

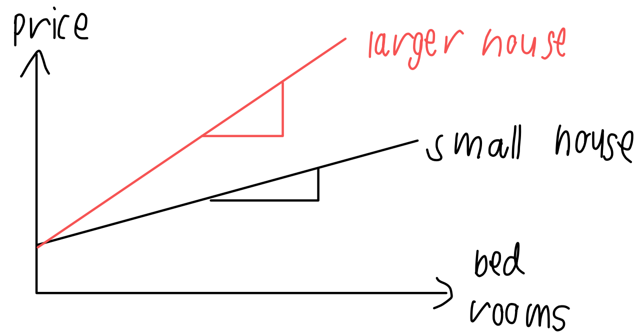
3 Models with Interaction Terms

⇒ use when the impact of one variable depends on the value (level) of another variables

Consider

$$price = \beta_0 + \beta_1 \underset{x_1}{sqrft} + \beta_2 \underset{x_2}{bdrms} + \beta_3 \overbrace{\underset{x_1 \cdot x_2}{sqrft \times bdrms}}^{x_3} + \beta_4 \underset{x_4}{bthrms} + u$$

where
  $price$ = housing price
  $sqrft$ = house size (square feet)
  $bdrms$ = number of bedrooms
  $bthrms$ = number of bathrooms



$$\frac{\partial\, price}{\partial\, bdrms} = \beta_2 + \beta_3\, sqrft$$

⇒ if $\beta_2 > 0$ then, an additional bedroom would increase price more for a larger house !

4 More on the Goodness-of-Fit and Selection of Regressors

- Adding more regressors ALWAYS improve fit → R^2 always ↑

∘ But we lose the "degree of freedom"
(d.f. = free data point used to estimate the parameter)
→ 1 data point is sacrificed every time we estimate a parameter
∘ using R^2 would not punish "having too many regressors"

∘ we use adjusted_R^2 or $\bar{R}^2$ when we want to punish adding too many regressors

$$R^2 = 1 - \frac{SSR}{SST} = 1 - \frac{SSR/\cancel{n}}{SST/\cancel{n}}$$

$$\text{adj } R^2 = 1 - \frac{SSR/(n-k-1)}{SST/(n-1)}$$

If we have more k, d.f. $= n-k-1$ ↓, $SSR/(n-k-1)$ ↑, adj_R^2 ↓

Using adjusted R-squared to choose between non-nested models (one model is not a subset of another).

Consider Model 1 None of these model are subset of another

$$\begin{aligned} \widehat{salary} &= \begin{array}{ccc} 830.63 & +0.0163 sales & +19.63 roe \\ (223.90) & (0.0089) & (11.08) \end{array} \\ n &= 209, \quad R^2 = 0.029, \quad \bar{R}^2 = 0.020 \end{aligned}$$

Consider Model 2 much superior performance

$$\begin{aligned} \log(\widehat{salary}) &= \begin{array}{ccc} 4.36 & +0.2751 \log(sales) & +0.0179 roe \\ (0.29) & (0.033) & (0.004) \end{array} \\ n &= 209, \quad R^2 = 0.282, \quad \bar{R}^2 = 0.275 \end{aligned}$$

→ 27.5% of variation in y is explained.
So, this model is better !

Multiple Regression Analysis with Qualitative Information:

1 Outline

- Describing qualitative information
- Using a single dummy independent variable
- Using dummy variables for multiple categories
- Interactions involving dummy variables
- A binary dependent variable (Y variable): The linear probability model

2 Describing Qualitative Information

- "Female" and "Married" are qualitative variable.
- We arbitarily assign a dummy variable to decribe them.

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$
$$married = \begin{cases} 1 & \text{if married} \\ 0 & \text{otherwise (of if single)} \end{cases}$$

TABLE 7.1

A Partial Listing of the Data in WAGE1.RAW

person	*wage*	*educ*	*exper*	*female*	*married*
1	3.10	11	2	1	0
2	3.24	12	22	1	1
3	3.00	11	2	0	0
4	6.00	8	44	0	1
5	5.30	12	7	0	1
. . .	. . .	. . .	. . .	. . .	. . .
525	11.56	16	5	0	1
526	3.50	14	5	1	0

3 Models with a single dummy independent variable

Consider

Assume that this is a PRF

$$wage = \beta_0 + \delta_0 female + \beta_1 educ + u. \quad \text{——} (1)$$

where

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

In this case, the δ_0 (Delta) notation is used to highlight the interpretation of the parameters multiplying dummy variables. In other cases, we can use any notation that is the most convenient.

1. $E(wage \mid female, educ) = E(\beta_0 + \delta_0 female + \beta_1 educ + u \mid female, educ)$ (pop. para (constant))

$= \beta_0 + \delta_0 female + \beta_1 educ + \underbrace{E(u \mid female, educ)}_{=0}$ ⇓ (MLR 1-4 are satisfied) = 0

$= \beta_0 + \delta_0 female + \beta_1 educ$

2. Thus

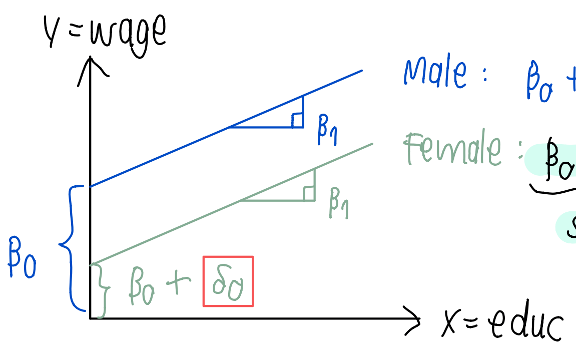
♀ : $E(wage \mid female = 1, educ) = \beta_1 + \delta_0(1) + \beta_1 educ = \beta_0 + \boxed{\delta_0} + \beta_1 educ$ — additional term for intercept

♂ : $E(wage \mid female = 0, educ) = \beta_1 + \delta_0(0) + \beta_1 educ = \beta_0 + \beta_1 educ$

$\delta_0 = E(wage \mid female = 1, educ) - E(wage \mid female = 0, educ)$

or $\delta_0 = E(wage \mid female, educ) - E(wage \mid male, educ)$

* given the same value of educ (same education level), δ_0 is the difference in the expected wage of females and males



Male : $\beta_0 + \beta_1 educ$

Female : $\underbrace{\beta_0 + \delta_0}_{\text{slope}} + \beta_1 educ$

(assume $\delta_0 < 0$)

→ By the way we model this regression function, "female" is going to give a constant impact on wage, regardless of the level of educ.

4 It is not possible to include all of the dummy alternatives in the same model (as long as there is an intercept in the model)

- If we include all alternatives of a dummy variable in the same model, we will face the "perfect collinearity" problem.

$$wage = \beta_0 X_0 + \delta_0 \, female + \beta_1 educ + \delta_1 male + u$$

X_0: intercept × 1; female (X_1); educ (X_2); male (X_3)

ID	female	male	X_0
1	1	0	1
2	1	0	1
3	0	1	1
4	0	1	1
⋮			
99			

For example:

$$X_0 = X_1 + X_3$$

$$1 = female + male$$
$$female = male + 1$$

or

If there are "n" catagories, we omit "1" catagories to avoid multicolinearity

$$1 = winter + spring + summer + fall$$
$$winter = 1 - spring - summer - fall$$

$$winter = \begin{cases} 1 & \text{if winter} \\ 0 & \text{otherwise} \end{cases}$$

$$spring = \begin{cases} 1 & \text{if spring} \\ 0 & \text{otherwise} \end{cases}$$

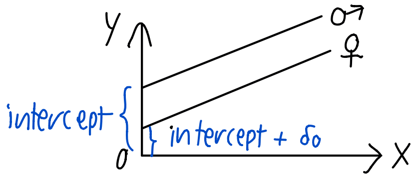
etc.

ID	winter	spring	summer	fall	X_0
1	1	0	0	0	1
2	0	1	0	0	1
3	0	0	1	0	1
4	0	0	0	1	1
⋮					

In this case, male

- At least one alternative has to be dropped. We treat the dropped alternative as the "BASE GROUP" or "BASELINE" or "BENCHMARK GROUP".

female $= \begin{cases} 1 & \text{if female} \\ 0 & \text{if male} \end{cases}$ male $= \begin{cases} 1 & \text{if male} \\ 0 & \text{if female} \end{cases}$




```
. regress lwage female male married educ exper
note: male omitted because of collinearity

      Source |       SS           df       MS      Number of obs   =       526
-------------+----------------------------------   F(4, 521)       =     75.27
       Model |  54.3265253         4  13.5816313   Prob > F        =    0.0000
    Residual |  94.0032262       521  .180428457   R-squared       =    0.3663
-------------+----------------------------------   Adj R-squared   =    0.3614
       Total |  148.329751       525  .28253286    Root MSE        =    .42477

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      female |  -.3251146   .0377061    -8.62   0.000    -.3991892     -.25104
        male |          0  (omitted)
     married |   .1380145   .0411197     3.36   0.001     .0572338    .2187953
        educ |   .0872644   .0071554    12.20   0.000     .0732075    .1013213
       exper |   .0076213   .0015314     4.98   0.000     .0046129    .0106297
       _cons |   .4690918   .1040575     4.51   0.000      .264668    .6735156
------------------------------------------------------------------------------
```

Female workers, are expected to have less wage compared to male workers

5 Using dummy variables for multiple categories

Case 1 We can use many dummy variables in the same model

Consider a model which includes 2 dummy variables– *female* and *married*.

$\begin{cases} 1 \text{ if female} \\ 0 \text{ if otherwise} \end{cases}$ $\begin{cases} 1 \text{ if married} \\ 0 \text{ if otherwise} \end{cases}$

$$\log(wage) = \beta_0 + \delta_0 female + \delta_1 married + \beta_1 educ + \beta_2 \exp er + \beta_3 \exp er^2 + \beta_4 tenure + \beta_5 tenure^2 + u.$$

```
regress lwage female married educ exper expersq tenure tenursq
```

Source	SS	df	MS
Model	65.6482326	7	9.37831895
Residual	82.6815188	518	.159616832
Total	148.329751	525	.28253286

Number of obs	= 526
F(7, 518)	= 58.76
Prob > F	= 0.0000
R-squared	= 0.4426
Adj R-squared	= 0.4351
Root MSE	= .39952

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.2901838	.0361121	-8.04	0.000	-.3611279	-.2192396
married	.0529219	.0407561	1.30	0.195	-.0271456	.1329894
educ	.0791547	.0068003	11.64	0.000	.0657952	.0925143
exper	.0269535	.0053258	5.06	0.000	.0164907	.0374163
expersq	-.0005399	.0001122	-4.81	0.000	-.0007603	-.0003196
tenure	.0312962	.0068482	4.57	0.000	.0178426	.0447499
tenursq	-.0005744	.0002347	-2.45	0.015	-.0010355	-.0001134
_cons	.4177837	.0988662	4.23	0.000	.2235557	.6120116

$\hat{\beta}$

2 δ_1 measures the impact of being married. (marriage premium)
But since $|t| < 1.96$ or $p > 0.05$, we do not reject H_0 of no impact

Comments:

1 δ_0 measures the expected difference between female and male workers given the same marital status and other factors

$$\frac{\partial \log(wage)}{\partial female} = \frac{\frac{1}{wage}\, d\,wage}{\partial female} = -0.29$$

$$\frac{100 \cdot \frac{1}{wage}\, d\,wage}{\partial female} = 100 \cdot -0.29$$

$$\frac{\%\,\Delta wage}{\partial female} = 29.02\,\%$$

- Female workers are expected to own less than male workers by 29.02%, holding other factors the same

	♀	♂
marr	marr fem	marrmale
sing	sing fem	singmale

Base case

Consider a model which includes dummy variables for each gender/marital status combination– *marrmale*, *marrfem* and *singfem*. (or singmale ← used as the basecase)

$$\log(wage) = \beta_0 + \delta_0 marrmale + \delta_1 marrfem + \delta_3 singfem + \beta_1 educ + \beta_2 exper + \beta_3 exper^2 + \beta_4 tenure + \beta_5 tenure^2 + u. \quad (8.1)$$

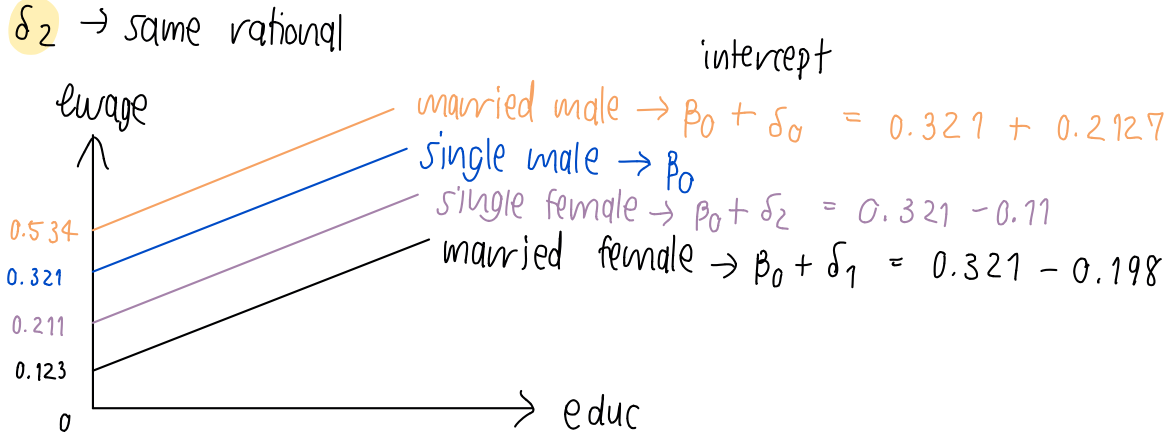
```
regress lwage marrmale marrfem singfem educ exper expersq tenure tenursq

      Source |       SS           df       MS      Number of obs   =       526
-------------+----------------------------------   F(  8,    517)  =     55.25
       Model |  68.3617623         8  8.54522029   Prob > F        =    0.0000
    Residual |  79.9679891       517  .154676961   R-squared       =    0.4609
-------------+----------------------------------   Adj R-squared   =    0.4525
       Total |  148.329751       525   .28253286   Root MSE        =    .39329

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    marrmale |   .2126757   .0553572     3.84   0.000      .103923    .3214284
     marrfem |  -.1982676   .0578355    -3.43   0.001     -.311889   -.0846462
     singfem |  -.1103502   .0557421    -1.98   0.048     -.219859   -.0008414
        educ |   .0789103   .0066945    11.79   0.000     .0657585     .092062
       exper |   .0268006   .0052428     5.11   0.000     .0165007    .0371005
     expersq |  -.0005352   .0001104    -4.85   0.000    -.0007522   -.0003183
      tenure |   .0290875    .006762     4.30   0.000     .0158031    .0423719
     tenursq |  -.0005331   .0002312    -2.31   0.022    -.0009874   -.0000789
       _cons |   .3213781    .100009     3.21   0.001     .1249041    .5178521
------------------------------------------------------------------------------
```

β

Comments:

- This regression is not the same as the previous one. It uses "singmale" as the base group. (The previous use male and single as 2 base group)

- δ_0 measures the expected different in wage of married male as compared with single males, holding other factors constant
- δ_1 measures the expected different in wage of married female as compared with single males, holding other factors constant
- δ_2 → same rational



Case 2 We can use dummy variables to represent multiple categories of a variable
Consider the relationship between law school rankings and starting salaries

$$\log(salary) = \beta_0 + \delta_0 top10 + \delta_1 r11\_25 + \delta_3 r26\_40 + \delta_4 r41\_60 + \beta_1 LSAT + \beta_2 GPA + \beta_3 \log(libvol) + \beta_4 \log(cost) + u.$$

where $top10$, $r11\_25$, $r26\_40$, $r41\_60$ would be equal to 1 when the variable $rank$ falls into the appropriate range.
** Rank below 60 would be the base case.

In many cases the "range of value" serve as a better explanatory variable than the "value" itself e.g. age may explain the model better if split into generations.

young 0-15
gen z 16-29
etc.

```
. regress lsalary top10 r11_25 r26_40 r41_60 LSAT GPA llibvol lcost
```

Source	SS	df	MS
Model	9.16538532	8	1.14567316
Residual	1.2109665	127	.009535169
Total	10.3763518	135	.076861865

Number of obs =	136
F(8, 127) =	120.15
Prob > F =	0.0000
R-squared =	0.8833
Adj R-squared =	0.8759
Root MSE =	.09765

lsalary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
top10	✓ .5393428	.053542	10.07	0.000	.4333927	.6452928
r11_25	✓ .4716199	.0390921	12.06	0.000	.3942637	.548976
r26_40	✓ .2790977	.0346972	8.04	0.000	.2104383	.3477571
r41_60	✓ .182382	.0283098	6.44	0.000	.126362	.238402
LSAT	.0060482	.0034919	1.73	0.086	-.0008616	.012958
GPA	.1305893	.0818678	1.60	0.113	-.0314122	.2925908
llibvol	.0725522	.0289213	2.51	0.013	.0153221	.1297824
lcost	.0249169	.0283224	0.88	0.381	-.031128	.0809619
_cons	8.363103	.4457314	18.76	0.000	7.481081	9.245125

the baseline is ranking 61th and worse

Comments:

rank	top10	r11_25	r26_40
1	1	0	0
2	1	0	0
3	1	0	0
⋮	⋮	⋮	⋮
10	1	⋮	⋮
11		1	⋮
12		1	⋮
⋮		⋮	⋮
25		1	⋮
26		⋮	1
⋮		⋮	⋮
40		⋮	1
⋮		⋮	

1. δ_0 measures the difference in expected log(salary) of a law-school graduate from a top 10 university compared to expected log(salary) of those who graduated from the school ranked 61th and worse

2. δ_1 → use the same rational (Top 11-25)