3 Differentiation: Basic Concepts

Differentiation is a technique that enable us to find out how a function changes when its argument changes.

3.1 New Vocabulary

Differentiation (n)	A technique for finding out how a function changes when its argument changes.
Differentiate (v)	A process of applying differentiation technique to a function
Derivative (n)	An outcome of differentiating a function.
Differentiable (adj)	When a derivative of a function can be obtained, that function is said to be differentiable.

3.2 Definition of Derivative

3.2.1 Mathematical definition

The derivative is given by the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: $\lim_{h \to 0}$ is for "the limit as h tends to zero".

According to the formula, $f(x)$ is **differentiable** at $x = c$ if $f'(c)$ exists; that is, if the limit that defines $f'(x)$ exists when $x = c$.

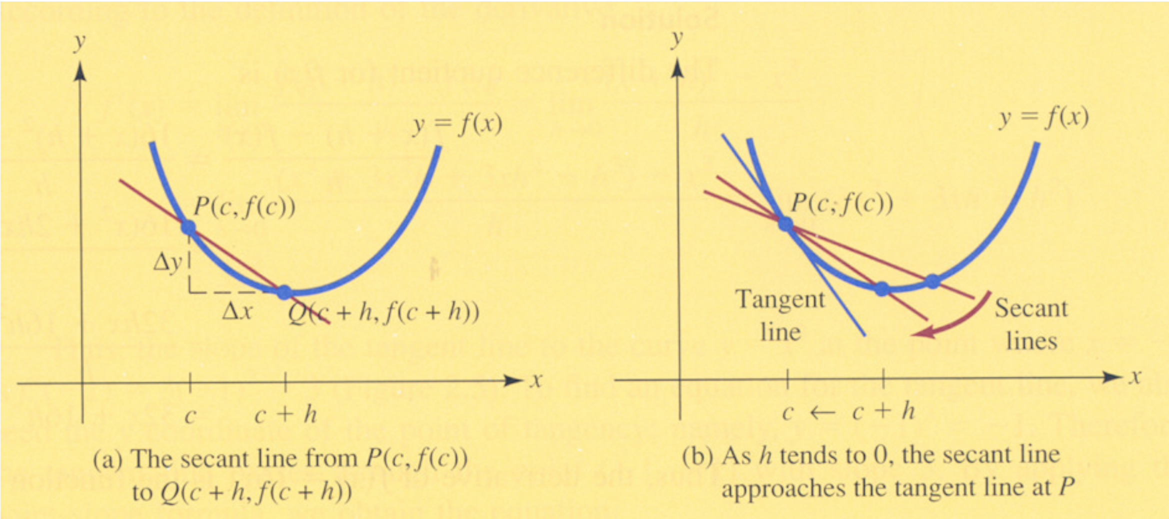


Figure 3.1: An overview of derivative formula

Hence, the derivative of a function $f(x)$ is the **SLOPE** of the **TANGENT** of the function $f(x)$ at point *x*.

- The mathematical symbols of a derivative of a function are normally $f'(x)$ or $\frac{dy}{dx}$ given $y = f(x)$.
- "Slop" measures the **STEEPNESS** of a line from left to right.
- "Tangent" of a curve at a particular point is a straight line which just touches the curve only at that point and only one point.

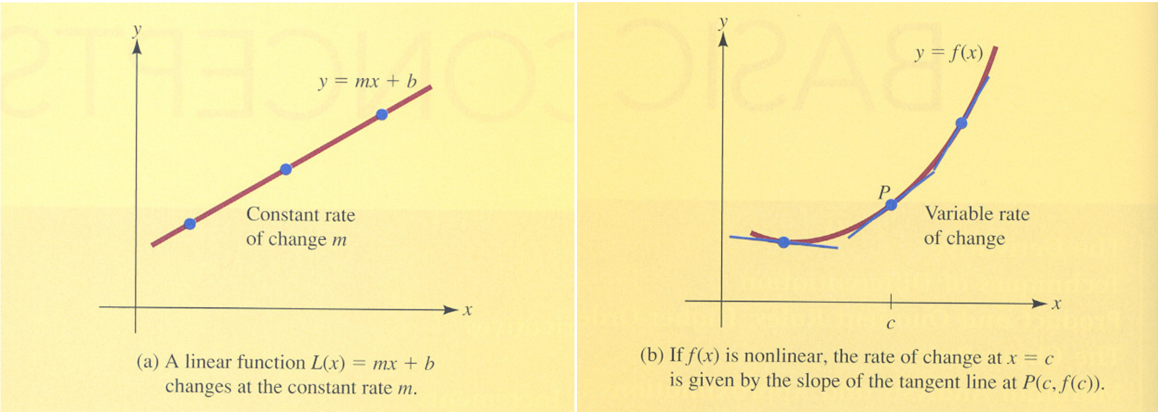


Figure 3.2: Definition of slope and tangent: (a) a linear function; (b) curve function.

3.2.2 Definition *vs*. Meaning

Slope as a Derivative: The slope of the tangent line to a curve $y = f(x)$ at the point $(c, f(c))$ is

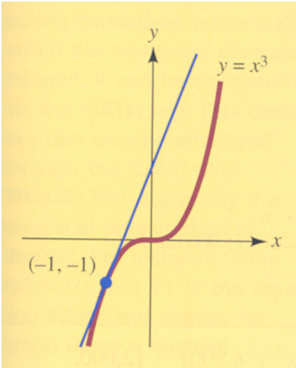
Instantaneous Rate of Change as a Derivative: The rate of change of $f(x)$ with respect to *x* when *x* = *c* is given by

Mariginal as a derivative: In economics, the derivatives are often described by the adjective "marginal." For example, the derivative of a cost function is called the marginal cost function.

3.2.3 Notation

Differentiate $y = f(x)$ with respect to ("w.r.t.") x equals to find:

- $f'(x)$ is pronounced "*f* dash *x*" or "*f* prime *x*"
- $\dfrac{dy}{dx}$ "dee *y* by dee *x*" or "dee *y*, dee *x*"
- $\dfrac{d}{dx}f(x)$ "dee by dee *x* of *f x*"
- $\dot{f}$ "*f* dot" A special case when *f* is differentiated w.r.t. time.
- $f'(c)$, $\left.\dfrac{dy}{dx}\right|_{x=c}$, $\left.\dfrac{d}{dx}f(x)\right|_{x=c}$ The derivative at $x = c$.



Ex. 1: Use ***<u>the definition of differentiation</u>*** to find the derivatives of the following functions.

(a) For $f(x) = x$, find $f'(1)$ and $f'(2)$. Ans: 1, 2

(b) For $f(x) = x^2$, find $f'(1)$, $f'(3)$ and $f'(-2)$ and the equation of the tangent at point (3,9).

Ans: $f'(x) = 2x$, 2, 6,-4, $y = 6x - 9$

(c) For $f(x) = x^3$, find $f'(x)$, $f'(-1)$ and the equation of the tangent where $x = -1$. Ans: $3x^2$, 3, $y = 3x + 2$

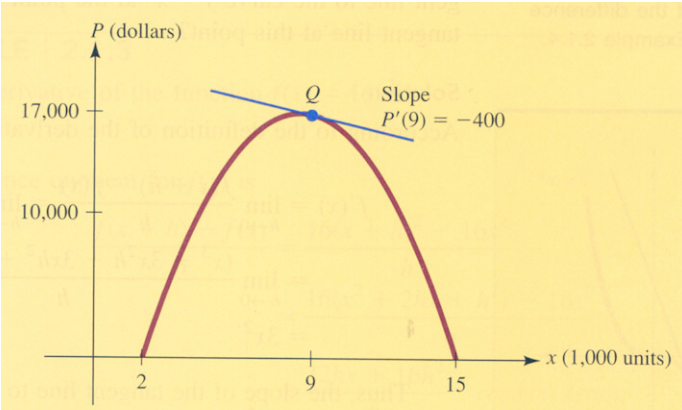
(d) For $f(x) = 9x^3 - 7x^2 + 15$, find $f'(x)$. Ans: $27x^2 - 14x$

Ex. 2: A manufacturer determines that when <u>*x* thousand units</u> of a particular commodity are produced, the profit generated will be

$$P(x) = -400x^2 + 6{,}800x - 12{,}000 \quad \text{dollars}$$

At what rate is profit changing with respect to the level of production *x* when 9,000 units are produced?

Ans: -400



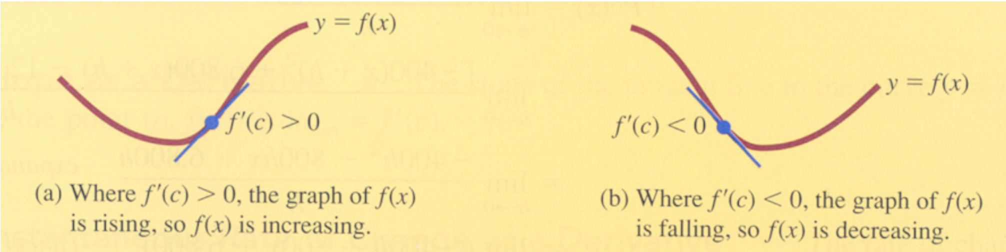
3.2.4 Significance of the Sign of the Derivative $f'(x)$

If the function *f* is differentiable at $x = c$, then

f is **increasing** at x = c if

and

f is **decreasing** at x = c if



(a) Where $f'(c) > 0$, the graph of $f(x)$ is rising, so $f(x)$ is increasing.

(b) Where $f'(c) < 0$, the graph of $f(x)$ is falling, so $f(x)$ is decreasing.

Figure 3.3: Increasing and decreasing functions

Ex. 3: Use the definition of differentiation to compute the derivative of $f(x) = \sqrt{x}$, then use the derivative to:

(a) Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point where $x = 4$.

(b) Find the rate at which $y = \sqrt{x}$ is changing with respect to x when $x = 1$.

Ans: $\frac{df}{dx} = \frac{1}{2\sqrt{x}}$, $f'(4) = \frac{1}{4}$, $y = \frac{1}{4}x + 1$, $\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2}$

3.3 Continuity of a Differentiable Function

If the function $f(x)$ is differentiable at $x = c$, then it is also continuous at point $x = c$.

However, a continuous function $f(x)$ is not always differentiable. A continuous function $f(x)$ will not be differentiable at $x = c$ if $f'(x)$ becomes infinite at $x = c$ or if the graph of $f(x)$ has a "shape" point at $P(c, f(c))$; that is, a point where the curve makes an abrupt change in direction.

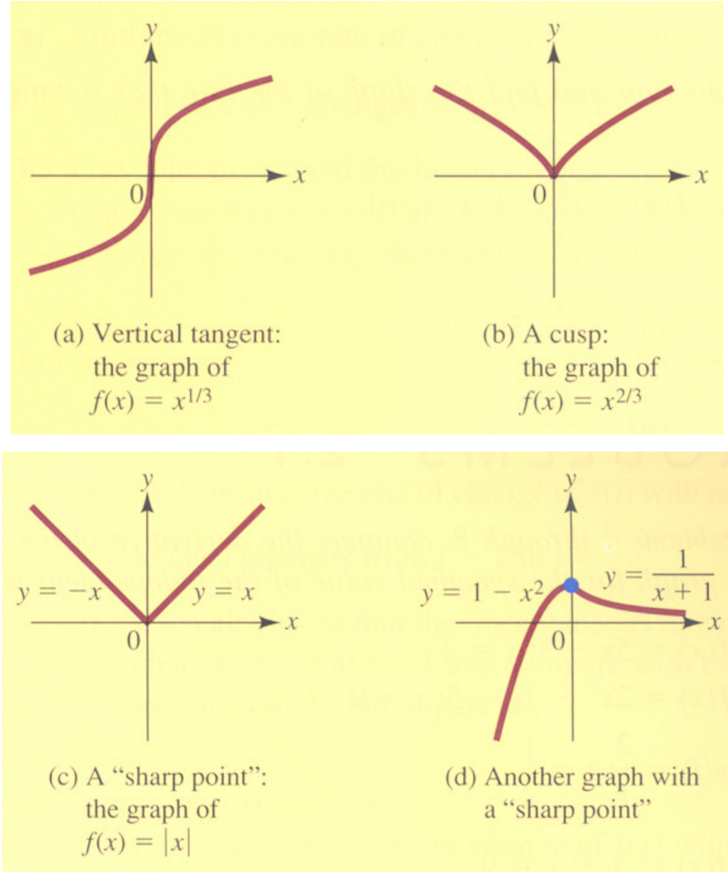
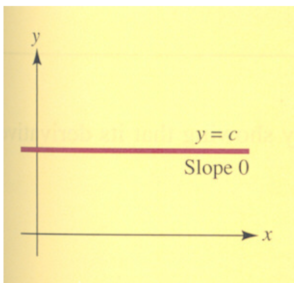


Figure 3.4: Examples of continuous functions that are not differentiable.

3.4 Differentiation Techniques

Finding derivatives using the definition could be very tedious. Instead there are simple formulas that can be used to help finding derivatives. These formulas can be derived using definition of derivative.

3.4.1 Constant Function Rule



For any constant c,

$$\frac{d}{dx}(c) =$$

The derivative of any constant function is zero.

(Prove using definition of derivative!!!)

3.4.2 Power Rule

For any real number *n*,

$$\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$$

The derivative of *x* to the power *n* is *n* times *x* to the power *n* minus 1.

(Prove using definition of derivative!!!)

Ex. 4: Find the derivatives of x, x^2 and x^3 [Ex. 1 (a)-(c)]

Ex. 5: Verify the power rule for the function $f(x) = \frac{1}{x^2} = x^{-2}$ by showing that its derivative is $f'(x) = -2x^{-3}$.

3.4.3 Constant Multiply Property

If *c* is a constant and $f(x)$ is differentiable, then

$$\frac{d}{dx}\left[c\,f(x)\right] =$$

The derivative of a constant times a differentiable function is the constant times the derivative of the function.

Ex. 6: Find the derivatives of the following functions.

(a) $f(x) = 9x^3$

Ans: $27x^2$

(b) $f(x) = 4\sqrt{x}$

Ans: $2/\sqrt{x}$

(c) $f(x) = -2x^{-1.3}$

Ans: $2.6x^{-2.3}$

(d) $f(x) = \frac{-7}{\sqrt{x}}$

Ans: $\frac{7}{2}x^{-\frac{3}{2}}$

3.4.4 Sum and Difference Property

If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] =$$

The derivative of the sum/difference of two differentiable functions is the sum/difference of the derivatives.

Ex. 7: Find the derivatives of the following functions.

(a) $f(x) = 9x^3 - 7x^2 + 15$ [Ex. 1(d)] Ans: $27x^2 - 14x$

(b) $f(x) = x^{-2} + 7$ Ans: $-2x^{-3}$

(c) $f(x) = 2x^5 - 3x^{-7}$ Ans: $10x^4 + 21x^{-8}$

(d) $f(x) = 5x^3 - 4x^2 + 12x - 3$ Ans: $15x^2 - 8x + 12$

3.4.5 Product Rule

If $f(x)$ and $g(x)$ are differentiable at *x*, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Ex. 9: Determine the derivative of the product of $f(x)$ and $g(x)$ by (i) expanding the expression into polynomials and (ii) using the product rule without expanding where

(a) $f(x) = x^2$ and $g(x) = x^3$ Ans: $5x^4$

(b) $f(x) = x - 1$ and $g(x) = 3x - 2$ Ans: $6x - 5$

(c) $f(x) = 5x + 3$ and $g(x) = x^2 - x + 1$ Ans: $15x^2 - 4x + 2$

Ex. 10: Find the derivative of $y(x) = 2x^{\frac{3}{2}}\left(x^3 - 2\right)$. Ans: $9x^{\frac{7}{2}} - 6x^{\frac{1}{2}}$

Ex. 11: For the curve $y = (2x+1)(2x^2 - x - 1)$,

(a) Find y'.

(b) Find an equation for the tangent line to the curve at the point where $x = 1$.

(c) Find all points on the curve where the tangent line is horizontal.

Ans: $y' = 12x^2 - 3$, $y = 9x - 9$, $\left(\frac{1}{2}, -2\right), \left(-\frac{1}{2}, 0\right)$

3.4.6 Quotient Rule

If $f(x)$ and $g(x)$ are differentiable at x, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

The derivative of the quotient of two functions is the bottom function times the derivative of the top function minus the top function times the derivative of the bottom function, all over the bottom function squared.

Ex. 12: Differentiate the quotient $Q(x) = \dfrac{x^2 - 5x + 7}{2x}$ by

(a) Dividing through first.

(b) Using the quotient rule. Ans: $\frac{1}{2} - \frac{7}{2x^2}$

Ex. 13: Differentiate the following rational functions

(a) $y(x) = \dfrac{3x}{x^2 + 2}$ Ans: $\dfrac{6 - 3x^2}{\left(x^2 + 2\right)^2}$

(b) $y(x) = \dfrac{3x - 2}{2x + 1}$ Ans: $\dfrac{7}{(2x + 1)^2}$

3.4.7 The Chain Rule

If $y = f(u)$ is differentiable function of *u* and $u = g(x)$ is in turn a differentiable function of *x*, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

A change in *x* causes a change in *u* and leads to a change in *y*. This is a composite function $y = f(g(x))$.

Ex. 15: Differentiate $y = (3x+1)^2$ by

(a) Expanding into polynomials.

(b) Chain rule where $y = f(u) = u^2$ and $u = g(x) = 3x+1$.

Ans: $18x + 6$

Ex. 16: Differentiate the following functions with respect to *x*.

(a) $y(x) = 4(2x^3 - 1)^7$ Ans: $168x^2(2x^3-1)^6$

(b) $y(u) = u^n$ and *u* is a function of *x* Ans: $\frac{d}{dx}(u^n) = n u^{n-1}\frac{du}{dx}$

3.4.8 General Power Rule

(This is a special case of the **chain rule**.)

$$\frac{d}{dx}(u^n) = n u^{n-1}\frac{du}{dx}$$

Ex. 17: Differentiate the following functions.

(a) $y(x) = \frac{1}{1+x^2}$ Ans: $\frac{-2x}{(1+x^2)}$

(b) $y(x) = \sqrt{2x-3}$ Ans: $\frac{1}{\sqrt{2x-3}}$

(c) $y(x) = (2x^4 - x)^3$ Ans: $3(2x^4-x)^2(8x^3-1)$

(d) $y(x) = \sqrt{x^2+3x+2}$ Ans: $\frac{2x+3}{2\sqrt{x^2+3x+2}}$

(e) $y(x) = \frac{1}{(2x+3)^5}$ Ans: $-\frac{10}{(2x+3)^6}$

Ex. 18: Differentiate the following functions. (Combined Rules)

(a) $y(x) = x^3(x^2+1)^4$ Hint: Use Product Rule then Chain Rule

Ans: $x^2(x^2+1)^3(11x^2+3)$

(b) $y(x) = \left(\dfrac{x}{x+\alpha}\right)^{\beta}$ Hint: Use Chain Rule then Quotient Rule

Ans: $\dfrac{\alpha\beta x^{\beta-1}}{(x+\alpha)^{\beta+1}}$

(c) $y(x) = 5\left(1+\sqrt{x^3+1}\right)^{25}$ Hint: Use Chain Rule twice.

Ans: $\dfrac{375}{2}\left(1+\sqrt{x^3+1}\right)^{24}(x^3+1)^{-\frac{1}{2}}x^2$

Ex. 19: Differentiate the function $y(x) = (3x+1)^4(2x-1)^5$ and simplify your answer. Then find all values of $x = c$ for which the tangent line to the graph of $f(x)$ at $(c, f(c))$ is horizontal.

Ans: $f'(x) = 2(3x+1)^3(2x-1)^4(27x-1)$, $c = -\dfrac{1}{3}, \dfrac{1}{2}$ or $\dfrac{1}{27}$.

Ex. 20: The manager of an appliance manufacturing firm determines that when blenders are priced at p dollars apiece, the number sold each month can be modeled by

$$D(p) = \frac{8{,}000}{p}.$$

The manager estimates that t months from now, the unit price of the blenders will be $p(t) = 0.06t^{\frac{3}{2}} + 22.5$ dollars. At what rate will the monthly demand for blenders $D(p)$ be changing 25 months from now? Will it be increasing or decreasing at this time?

Ans: $\left.\dfrac{dD}{dt}\right|_{\substack{t=25\\p=30}} = -4$

3.5 Relative and Percentage Rate of Change:

The relative rate of change of a quantity $Q(x)$ with respect to *x* is given by the ratio

$$\text{Relative rate of change of } Q(x) = \frac{Q'(x)}{Q(x)} =$$

The corresponding **percentage rate of change** of $Q(x)$ with respect to *x* is

$$\text{Percentage rate of change of } Q(x) =$$

Ex. 21: The gross domestic product (GDP) of a certain county was $N(t) = t^2 + 5t + 106$ billion dollars *t* years from 1995.

(a) At what rate was the GDP changing with respect to time in 2005?

(b) At what percentage rate was the GDP changing with respect to time in 2005?

Ans: $N'(t) = 2t + 5$, 25 billion dollars per year, 9.77% per year

3.6 The Higher-Order Derivative

It is possible that the rate of change of a function is itself a rate of change. For example, the acceleration of a car is the rate of change with respect to time of its velocity, which in turn is the rate of change with respect to time of its position.

$y = f(x)$

$y' = \dfrac{dy}{dx}$ is the derivative of *y*.

$y'' = \dfrac{dy'}{dx} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} \neq \left(\dfrac{dy}{dx}\right)^2$ is the second derivative of *y*.

$y''' = \dfrac{dy''}{dx} = \dfrac{d}{dx}\left\{\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)\right\} = \dfrac{d^3y}{dx^3} \neq \left(\dfrac{dy}{dx}\right)^3$ is the third derivative of *y*.

Ex. 22: Find the second derivative of the function $f(x) = \dfrac{3x-2}{(x-1)^2}$.

Ans: $f'(x) = \dfrac{1-3x}{(x-1)^3}, f''(x) = \dfrac{6x}{(x-1)^4}$

Ex. 23: Determine $y', y'', y''', y^{(4)}$ and $y^{(5)}$, for

(a) $y = f(x) = x^4 + x^3 + x^2 + x + 1$

(b) $y = f(x) = \dfrac{1}{x}$