Introduction to Binomial Trees II

Chapter 11

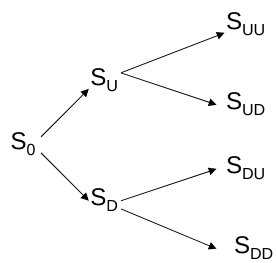
Outline

- Binomial tree
- Valuing American options
- The distribution of the underlying asset and u and d
- Delta

Multi-stage Binomial Option Pricing

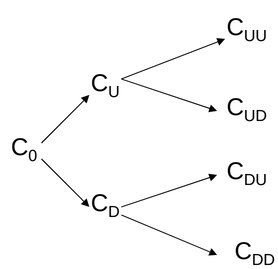
<u>Just solve series of one-period models. Work backwards!</u>

<u>Step 1</u>: Work out all stock prices. In two-period model, there are 2+4 = 6 of them.



Multi-stage Binomial Option Pricing

<u>Step 2</u>: Work out all C_{XX} at the termination (4 of them). Just solve $C_{XX}= \max(0,S_{XX}-X)$

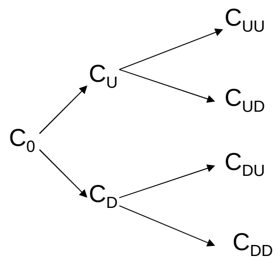


<u>Step 3</u>: Use one-period model to solve for C_X (2 of them)

<u>Step 4</u>: Use one-period model to solve for C_0 .

American Option Pricing

Step 2: Work out all C_{XX} at the termination (4 of them). Just solve C_{XX}= max(0,S_{XX}-X)



Step 3: Use one-period model to solve for C_X and compare C_X and the intrinsic value of immediate exercise. The higher value of the two is the option price

Step 4: Use one-period model to solve for C_0 .

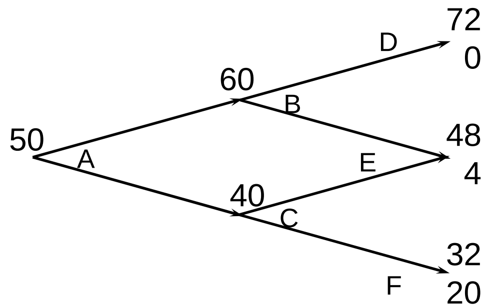
Option Pricing: Two & n Subperiods

$$C_u = Max(S_u - K, e^{-rT/2}[p \cdot C_{uu} + (1-p) \cdot C_{ud}])$$
$$C_d = Max(S_d - K, e^{-rT/2}[p \cdot C_{ud} + (1-p) \cdot C_{dd}])$$

For n subperiods

$$C_{u..n} = Max(S_{u..n} - K, e^{-rT/n}[p \cdot C_{uu..n} + (1-p) \cdot C_{ud..n}])$$
$$C_{d..n} = Max(S_{d..n} - K, e^{-rT/n}[p \cdot C_{ud..n} + (1-p) \cdot C_{dd..n}])$$

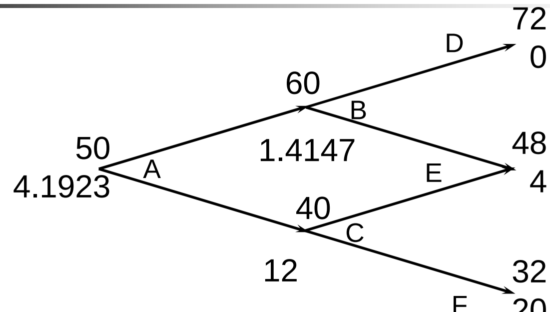
A Put Option Example; K=52; T=2; r=5%



$$u = 60/50 = 1.2; d = 40/50 = 0.8$$

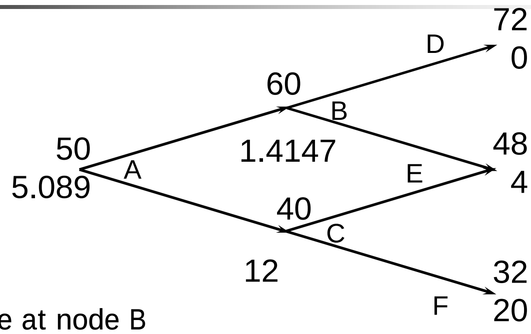
$$p = \frac{e^{rT/2} - d}{u - d} = \frac{e^{0.05\times 1} - 0.8}{1.2 - 0.8} = 0.6282$$

A Put Option Example; K=52



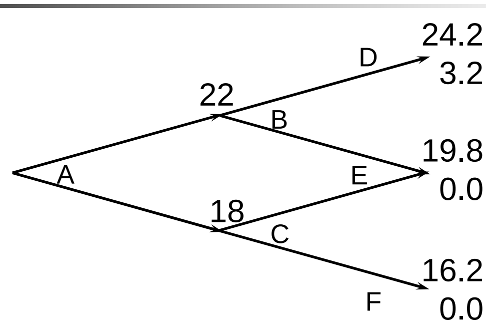
- Value at node C
  = Max(52-40, $e^{-0.05\times 1}$(0.6282×4 + 0.3718×20)
  = Max(12, 9.4636) =12
- So the option is exercised at node C

A Put Option Example; K=52



- Value at node B
  $= \text{Max}\ (0,\ e^{-0.05\times 1}(0.6282\times 0\ +\ 0.3718\times 4) = 1.4147$
- Value at node A
  $= \text{Max}\ (2,\ e^{-0.05\times 1}(0.6282\times 1.4147\ +\ 0.3718\times 12) = 5.089$

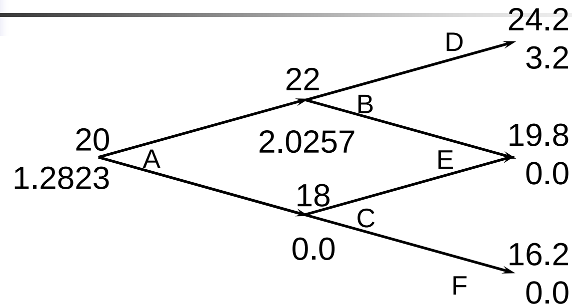
Valuing a Call Option K=21, T=0.5 r=12%



p for this binomial tree is

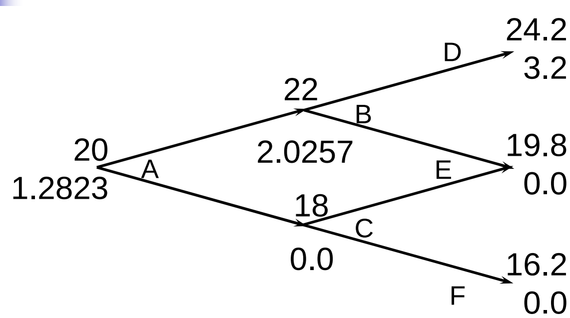
$$p = \frac{e^{rT/2} - d}{u - d} = \frac{e^{0.12\times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing a Call Option



- Value at node C
  $= \text{Max}\ (0,\ e^{-0.12\times0.25}(0.6523\times0 + 0.3477\times0))=0$
- Value at node B
  $= \text{Max}\ (1,\ e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0)) =$ 2.0257

Valuing a Call Option



- Value at node A
  $= \text{Max}\ (0,\ e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0))=1.2823$

Choosing *u* and *d*

- The stochastic distribution of the underlying asset return is usually estimated as a particular statistic distribution such as the normal distribution or the mean-reverting distribution
- The up or down move of the underlying asset in a Binomial tree is determined by the volatility of the underlying asset returns

Choosing *u* and *d*

If the underlying asset returns follows a normal distribution, that is the price follows a log-normal distribution then one way of matching the volatility is to set

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma \sqrt{\Delta t}}$$

where σ is the volatility of the return of the underlying asset and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein.

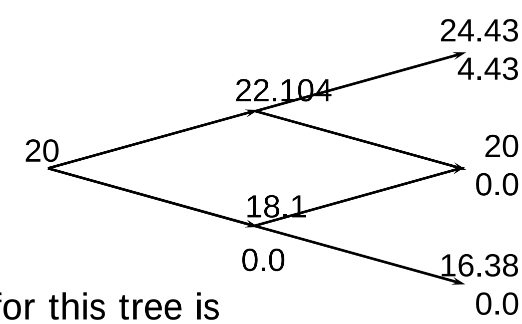
Example

Example: Use the two-step Binomial tree to calculate the price of a European call option with K=20 T=0.5, and r=6%. The underlying asset return has a volatility = 0.2.

$$u = e^{0.2\sqrt{0.25}} = 1.1052$$

$$d = 1/u = e^{-0.2\sqrt{0.25}} = 0.905$$

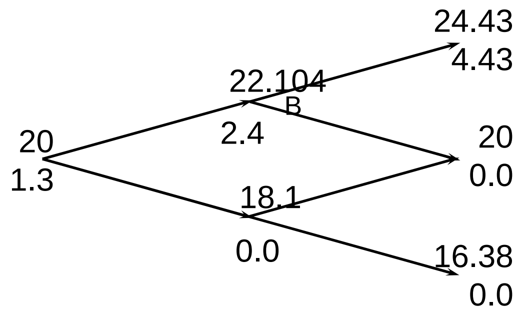
Example



p for this tree is

$$p = \frac{e^{0.06\times 0.25} - 0.905}{1.1052 - 0.905} = 0.55$$

Example



Value at B is $e^{-0.06\times0.25}(0.55\times4.43 + 0.45\times0)=2.4$

Value of the option is $e^{-0.06\times0.25}(0.55\times2.4 + 0.45\times0)=1.3$

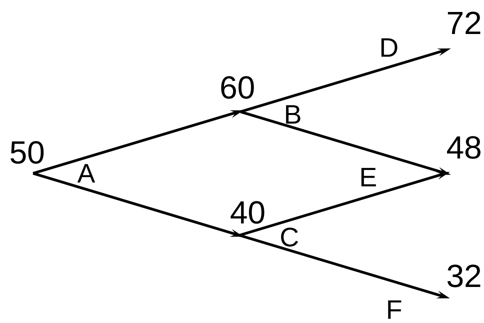
Concept check (r=5%)

A 2-year put option with K=55. What should be the price of this put option?

G) 6.79

Y) 4.8

R) 5.3

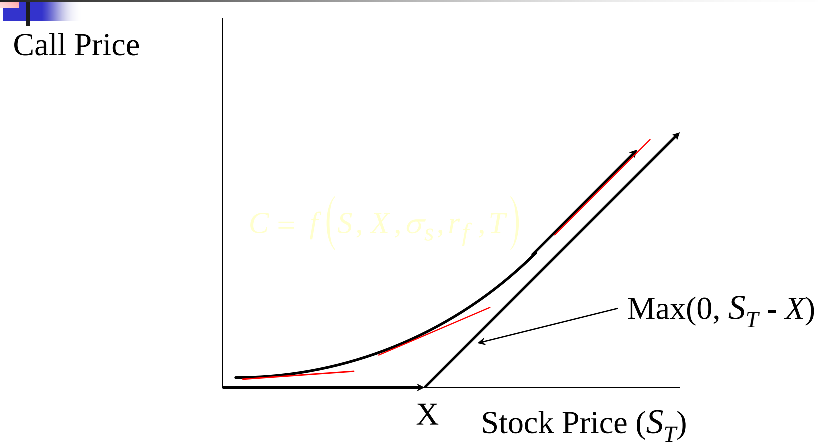


Delta

- Delta (Δ)

  1. Delta is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
  2. The dynamic replicating portfolio of a call option consists of long Δ number of stocks and borrowing money
  3. Delta is the slope of the line tangent to the call option pricing function with the underlying asset

Call options



A call option

A 3-month call option on the stock has a strike price of 21.

Stock Price = \$22
Option Price = \$1

Stock price = \$20
Option Price=?

Stock Price = \$18
Option Price = \$0

Replicating a call option pay off

- Δ = (1-0)/(22-18) = 0.25
- Consider a portfolio long 0.25 shares and borrow PV(4.5)

0.25*22-4.5 = 1

$18 * 0.25 - 4.5 = 0$

- We can replicate the pay off of a call option using a portfolio with Δ in the underlying stock and borrowing PV of the (Δ *Su-C_u)

Delta

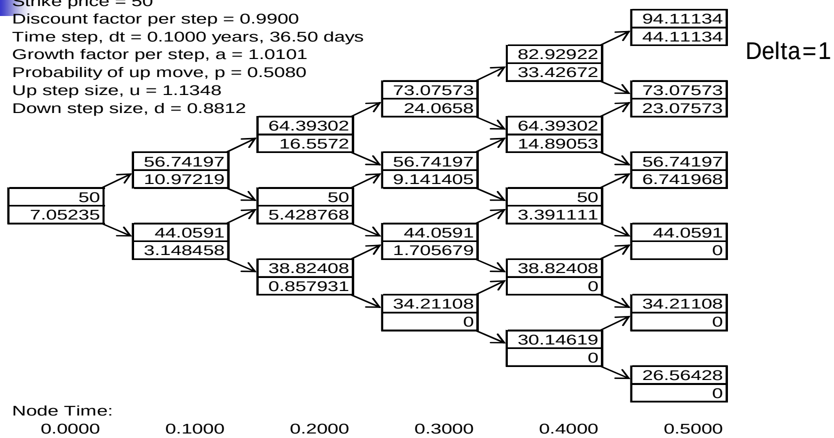
- The dynamic replicating portfolio of a call option consists of long Δ of stocks and borrow some money
- The value of Δ varies from node to node

Delta

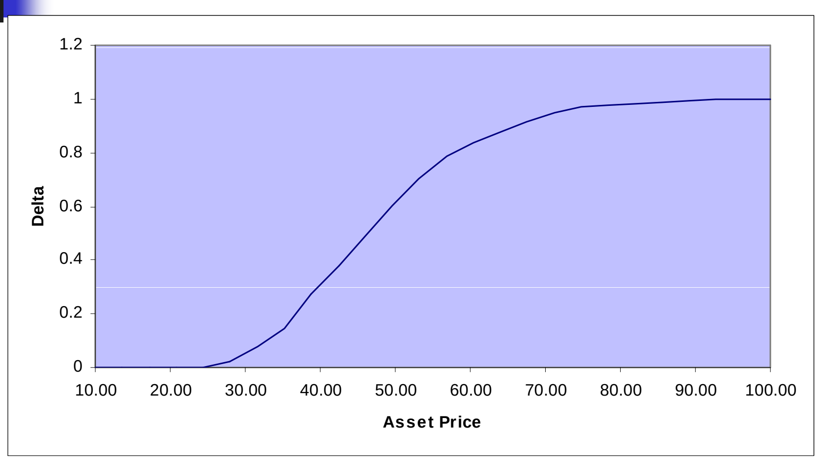
- Delta of a call option is between 0 and 1
- The more a call option is in the money the higher the delta is
- At the extreme, if it is certain that the option would expire in the money delta would be 1, since a replicating portfolio would need to invest in 1 share

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Values in red are a result of early exercise.

Strike price = 50
Discount factor per step = 0.9900
Time step, dt = 0.1000 years, 36.50 days
Growth factor per step, a = 1.0101
Probability of up move, p = 0.5080
Up step size, u = 1.1348
Down step size, d = 0.8812



Delta of a call option



Example: American Put Option

S_0 = 50; K = 50; r =10%; σ = 40%;

T = 5 months = 0.4167;

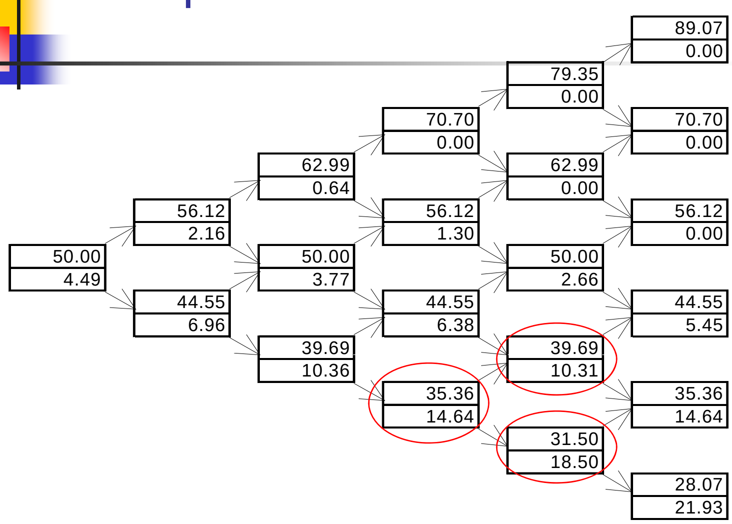
Δt = 1 month = 0.0833

The parameters imply

$u = 1.1224;\; d = 0.8909;$

$a = 1.0084;\; p = 0.5076$

Example



Concept check

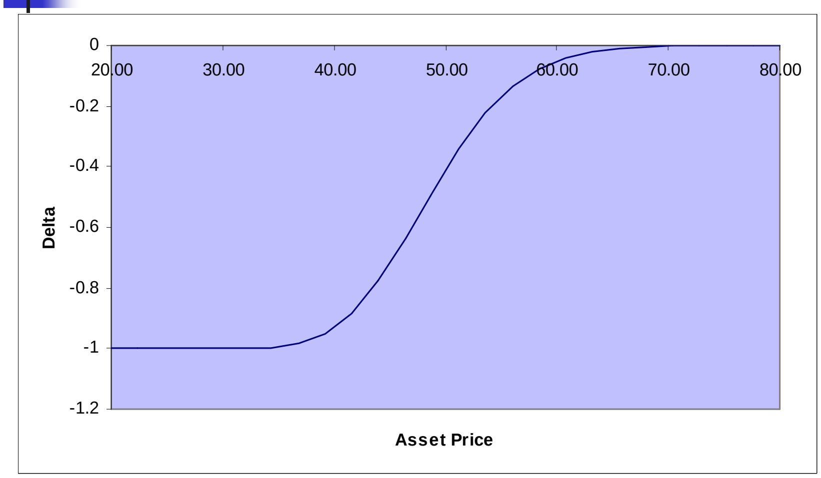
- What is the range of the delta of a European put option?

G) -1 and 1

Y) -1 and 0

R) 0 and 1

Delta of a put option



The Probability of an Up Move

$$p = \frac{a-d}{u-d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk-free rate

$a = 1$ for a futures contract

Summary

- Binomial tree
- Valuing American options
- The distribution of the underlying asset and u and d
- Delta