

EE325 Section 1 HW 2 Due Thursday February 20th (23:00 hr.), 2020

Use 4 decimal places for numerical answers

1. In Table 1.a. X_i is total microeconomics exam point (total points are 100) and Y_i is GPA of each student.

$$\bar{X} = \frac{63 + 72 + 78 + 81 + 87 + 75 + 75 + 90}{8} = 77.625$$

Table 1.a

Student	Y_i	X_i	$y_i - \bar{y}$	$x_i - \bar{x}$	\hat{u}_i
1	2.8	63	-0.4125	-14.625	0.0857
2	3.4	72	0.1875	-5.625	0.3791
3	3	78	-0.2125	0.375	-0.2253
4	3.5	81	0.2875	3.375	0.1725
5	3.6	87	0.3875	9.375	0.0681
6	3.0	75	-0.2125	-2.625	-0.1231
7	2.7	75	-0.5125	-2.625	-0.4231
8	3.7	90	0.4875	12.375	0.0659

1.1 Now consider the two-variable $Y_i = \beta_0 + \beta_1 X_i + u_i$, $u_i \sim NIID(0, \sigma^2)$. Use OLS to find the estimator of β_0 and β_1 . (Note: *NIID* = Normally, Identically, and Independently Distributed).

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} = \frac{(63 - 77.625)(2.8 - 3.2125) + (72 - 77.625)(3.4 - 3.2125) + \dots + (90 - 77.625)(3.7 - 3.2125)}{(-14.625)^2 + (-5.625)^2 + \dots + (12.375)^2}$$

$$= \frac{17.4375}{511.875} = 0.0341$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = 3.2125 - (0.0341)(77.625)$$

$$\hat{\beta}_0 = 0.5681$$

1.2 For each observation i , find \hat{Y}_i and \hat{u}_i . Show that $\sum_{i=0}^N \hat{u}_i = 0$.

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \quad \hat{u}_i = y_i - \hat{Y}_i$$

$$\hat{Y}_i = 0.5681 + 0.0341 X_i \quad \hat{u}_i = y_i - 0.5681 - 0.0341 X_i$$

$$\sum_{i=0}^{10} \hat{u}_i = (2.8 - 0.5681 - 0.0341(63)) + (3.4 - 0.5681 - 0.0341(72)) + \dots + (3.7 - 0.5681 - 0.0341(90))$$

$$= -0.0002 \approx 0$$

↑ This is NOT zero because β_1 and β_0 were rounded

$$SST = \sum_i (x_i - \bar{x})^2 = 551.875 \quad | \quad \sum x_i^2 = 48,717$$

$$SSR = \sum (\hat{y} - \bar{y})^2 = 1.02875$$

$$\sigma^2 = \text{Var}(u_i | X_i) = E[(u_i - E(u_i))^2] = E(u_i^2)$$

1.3 Find $\text{var}(\hat{u}_i)$, $\text{var}(\hat{\beta}_0)$, $\text{var}(\hat{\beta}_1)$

$$\text{var}(\hat{u}_i) = \sigma^2 = \frac{SSR}{n-2} = \frac{1.02875}{8-2} = 0.1715$$

$$\text{var}(\hat{\beta}_0) = \frac{\sigma^2 n^{-1} \sum_{i=1}^n x_i^2}{SST} = \frac{(0.1715)(48717)}{(8)(551.875)} = 2.0398$$

$$\text{var}(\hat{\beta}_1) = \frac{\sigma^2}{SST} = \frac{0.1715}{551.875} = 0.00033$$

2. Data is listed in the table

X_i	Y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	\hat{u}_i
10	0	-10	-9.1	-0.1455
12	2	-8	-7.1	0.0636
14	5	-6	-4.1	1.2727
16	6	-4	-3.1	0.4818
18	7	-2	-2.1	-0.3091
22	10	2	0.9	-0.8909
24	10	4	0.9	-2.6818
26	15	6	5.9	0.5273
28	16	8	6.9	-0.2636
30	20	10	10.9	1.9455

2.1 From the simple regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$, $u_i \sim NIID(0, \sigma^2)$. Find estimators of

β_0 and β_1 from the OLS method and interpret the meaning.

$$\bar{x} = \frac{10+12+14+16+18+22+24+26+28+30}{10} = 20$$

$$\bar{y} = \frac{0+2+5+6+7+10+10+15+16+20}{10} = 9.1$$

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} = \frac{(-10)(-9.1) + (-8)(-7.1) + \dots + (10)(10.9)}{(-10)^2 + (-8)^2 + \dots + (10)^2} = \frac{394}{440} = 0.8955$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = 9.1 - 0.8955(20)$$

$$\hat{\beta}_0 = -8.81$$

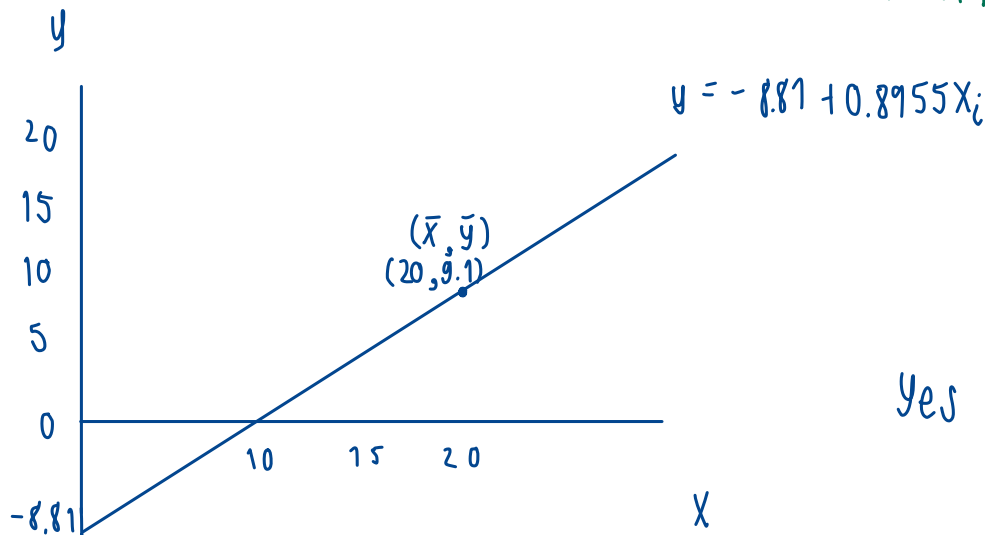
2.2 Find the value of \hat{Y}_i and \hat{u}_i . Show that $\sum_{i=0}^N \hat{u}_i = 0$.

$$\begin{aligned}\hat{y}_i &= \beta_0 + \beta_1 x_i & u_i &= y_i - \hat{y}_i \\ &= -8.81 + 0.8955 x_i & &= y_i + 8.81 - 0.8955 x_i\end{aligned}$$

$$\begin{aligned}\sum \hat{u}_i &= (0 + 8.81 - 0.8955(10)) + (2 + 8.81 - 0.8955(12)) + \dots + (20 + 8.81 - 0.8955(30)) \\ &= 0\end{aligned}$$

2.3 Plot graph and draw regression line. Does the line pass (\bar{X}, \bar{Y}) ?

$$\begin{aligned}y &= -8.81 + 0.8955(20) \\ y &= 9.1\end{aligned}$$



2.4 If $X_i = 16$, what is the predicted Y ?

$$\begin{aligned}E(y) &= -8.81 + 0.8955(16) \\ &= 5.518\end{aligned}$$

2.5 Find $\text{var}(\hat{u}_i)$, $\text{var}(\hat{\beta}_0)$, $\text{var}(\hat{\beta}_1)$

$$\begin{aligned}\text{SSR} &= \sum (y - \bar{y})^2 = 366.9 \\ \text{SST} &= \sum (x - \bar{x})^2 = 440\end{aligned}$$

$$\text{var}(\hat{u}_i) = \frac{\text{SSR}}{n-2} = \frac{366.9}{10-2} = 45.8625$$

$$\text{var}(\hat{\beta}_0) = \frac{6^2 n^{-1} \sum_{i=1}^n x_i^2}{\text{SST}} = \frac{(45.8625)(4440)}{10(440)} = 46.2794$$

$$\text{var}(\hat{\beta}_1) = \frac{6^2}{\text{SST}} = \frac{46.8625}{440} = 0.104233$$

3. Consider the below regression function:

$$Y_i = \beta_1 X_i + u_i,$$

Where $u_i \sim NIID(0, \sigma^2)$. Find an OLS estimator of β_1 . Then, provide a proof that this is an unbiased estimator. Please state the SLR assumption(s) when used.

$$y_i = \beta_1 x_i + u_i$$

$$\text{argmin } \sum_i (y_i - \tilde{\beta}_1 x_i)^2$$

F.O.C. w.r.t. $\tilde{\beta}_1$

$$\frac{\partial \sum_i (y_i - \tilde{\beta}_1 x_i)^2}{\partial \tilde{\beta}_1} = 0$$

$$\sum_i 2(y_i - \tilde{\beta}_1 x_i)(-x_i) = 0$$

$$\sum_i x_i (y_i - \tilde{\beta}_1 x_i) = 0 \quad (-2)$$

$$\sum_i x_i y_i - \sum_i \tilde{\beta}_1 x_i^2 = 0$$

$$\sum_i x_i y_i = \sum_i \tilde{\beta}_1 x_i^2$$

$$\hat{\beta}_1 = \frac{\sum_i x_i y_i}{\sum_i x_i^2}$$

Substitute

$$y_i = \beta_1 x_i + u_i$$

$$\hat{\beta}_1 = \frac{\sum_i x_i (\beta_1 x_i + u_i)}{\sum_i x_i^2}$$

$$\hat{\beta}_1 = \frac{\sum_i \beta_1 x_i^2}{\sum_i x_i^2} + \frac{\sum_i u_i x_i}{\sum_i x_i^2}$$

$$E(\hat{\beta}_1) = E(\beta_1) + E\left(\frac{\sum_i u_i x_i}{\sum_i x_i^2}\right) \quad \text{by SLR4 } E(u_i | x_i) = 0$$

$$E(\hat{\beta}_1) = \beta_1$$