EE320 (2/2012)

INTRODUCTORY MATHEMATICAL ECONOMICS

MATRIX ALGEBRA AND ITS APPLICATION

(Part 2 – Matrix Applications)

Topics

- Partial market equilibrium
- General market equilibrium
- Excise tax and market equilibrium
- Simple macroeconomic model
- IS-LM model
- Input-output model

Partial Market Equilibrium (1)

- Given the system of equations

$$Q_d = 24 - 4P \qquad \text{-- (1)}$$

$$Q_s = -27 + 13P \qquad \text{-- (2)}$$

$$Q_d = Q_s \qquad \text{-- (3)}$$

- Write the system in matrix form $AX = d$:

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -13 \\ 1 & -1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 24 \\ -27 \\ 0 \end{bmatrix}$$

Partial Market Equilibrium (2)

- Using Cramer's rule to find Q* and P*

➢ Answer: $P^* = 3$, $Q^* = 12$

General Market Equilibrium (1)

- Market for good 1

$$Q_{d1} - Q_{s1} = 0 \quad (1)$$
$$Q_{d1} = a_0 + a_1 P_1 + a_2 P_2 \quad (2)$$
$$Q_{s1} = b_0 + b_1 P_1 + b_2 P_2 \quad (3)$$

- Market for good 2:

$$Q_{d2} - Q_{s2} = 0 \quad (4)$$
$$Q_{d2} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \quad (5)$$
$$Q_{d2} = \beta_0 + \beta_1 P_1 + \beta_2 P_2 \quad (6)$$

•Write the above system of equations in the matrix form:

Define $c_i \equiv a_i - b_i$ and $\gamma_i \equiv \alpha_i - \beta_i$

$$\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -c_0 \\ -\gamma_0 \end{bmatrix}$$

General Market Equilibrium (2)

- Find the equilibrium prices and quantities for good 1 and good 2.

Define: $c_i \equiv a_i - b_i$

$$\gamma_i \equiv \alpha_i - \beta_i$$

➢ Answer:

$$\boxed{P_1^* = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1}}$$

$$\boxed{P_2^* = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}}$$

Excise Tax and Market Equilibrium: Specific Tax (1)

- Suppose a specific tax t baht per unit is imposed on consumer.

$$Q_D = Q_S \qquad (1)$$

$$Q_D = a - b(P + T) \qquad (2)$$

$$Q_S = -c + dP \qquad (3)$$

- Write the above system of the equations in the matrix form:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a - bT \\ -c \end{bmatrix}$$

Excise Tax and Market Equilibrium: Specific Tax (2)

- Find the new equilibrium prices and quantity.
- Answer:

$$\boxed{P^* = \frac{a+c+dT}{b+d} = P_D^{'}}$$

$$\boxed{P^T = P_D^{'} - T = \frac{a+c-bT}{b+d} = P_S^{'}}$$

$$\boxed{Q^* = \frac{ad-bc-bdT}{b+d}}$$

Excise Tax and Market Equilibrium: Ad Varolem Tax (1)

- Suppose an *t% tax* is imposed on consumer.

$$Q_D = Q_S \qquad (1)$$

$$Q_D = a - bP^T \qquad (2)$$

$$Q_S = -c + dP \qquad (3)$$

$$P^T = (1+t)P \qquad (4)$$

- Write the above system of the equations in the matrix form:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & 1 & -d & 0 \\ 0 & 0 & -(1+t) & 1 \end{bmatrix} \qquad X = \begin{bmatrix} Q_d \\ Q_s \\ P \\ P^T \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ a \\ -c \\ 0 \end{bmatrix}$$

Excise Tax and Market Equilibrium: Specific Tax (2)

- Find the new equilibrium prices and quantity.
- Answer:

$$\boxed{P^* = \frac{a+c}{b(1+t)+d} = P_S^{'}}$$

$$\boxed{P^t = (1+t)P_s^{'} = \frac{(1+t)(a+c)}{b(1+t)+d} = P_D^{'}}$$

$$\boxed{Q^* = \frac{ad-(1+t)bc}{b(1+t)+d}}$$

Simple Macroeconomic Model (1)

- Keynesian model:

$$Y = C + I_0 + G_0 + X_0 - M$$

$$C = a + bY_d$$

$$M = mY_d$$

$$Y_d = Y - T$$

$$T = tY$$

- Reduce the above system into:

Simple Macroeconomic Model (2)

- Write the above system in the matrix form, and find the equilibrium national income.

$$\begin{bmatrix} 1+m(1-t) & -1 \\ -b(1-t) & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 + X_0 \\ a \end{bmatrix}$$

- Solutions:

$$\boxed{Y^* = \frac{a + I_0 + G_0 + X_0}{1-b(1-t)+m(1-t)}}$$

$$\boxed{C^* = \frac{a[1+m(1-t)]+b(1-t)(I_0+G_0+X_0)}{1-b(1-t)+m(1-t)}}$$

IS-LM Model (1)

- Commodity Market:

$$Y = C + I + G_0$$

$$C = a + bY_d \qquad (a > 0,\ 0 < b < 1)$$

$$I = I_0 - ir \qquad (I_0,\ i > 0)$$

$$Y_d = Y - T \qquad \text{where } T = tY \qquad (0 < t < 1)$$

$\Rightarrow$ $Y = a + b(1-t)Y + I_0 - ir + G_0$: IS

- Money Market:

$$M^S = M_0$$

$$M^D = kY - hr \qquad (k,\ h > 0)$$

$\Rightarrow$ $M_0 = kY - hr$: LM

- Write a matrix form of the above IS-LM equations:

IS-LM Model (2)

- Use Cramer's rule to find the equilibrium national income and interest rate.

$$\begin{bmatrix} 1-b(1-t) & i \\ k & -h \end{bmatrix} \begin{bmatrix} Y \\ r \end{bmatrix} = \begin{bmatrix} a+I_0+G_0 \\ M_0 \end{bmatrix}$$

- Solutions:

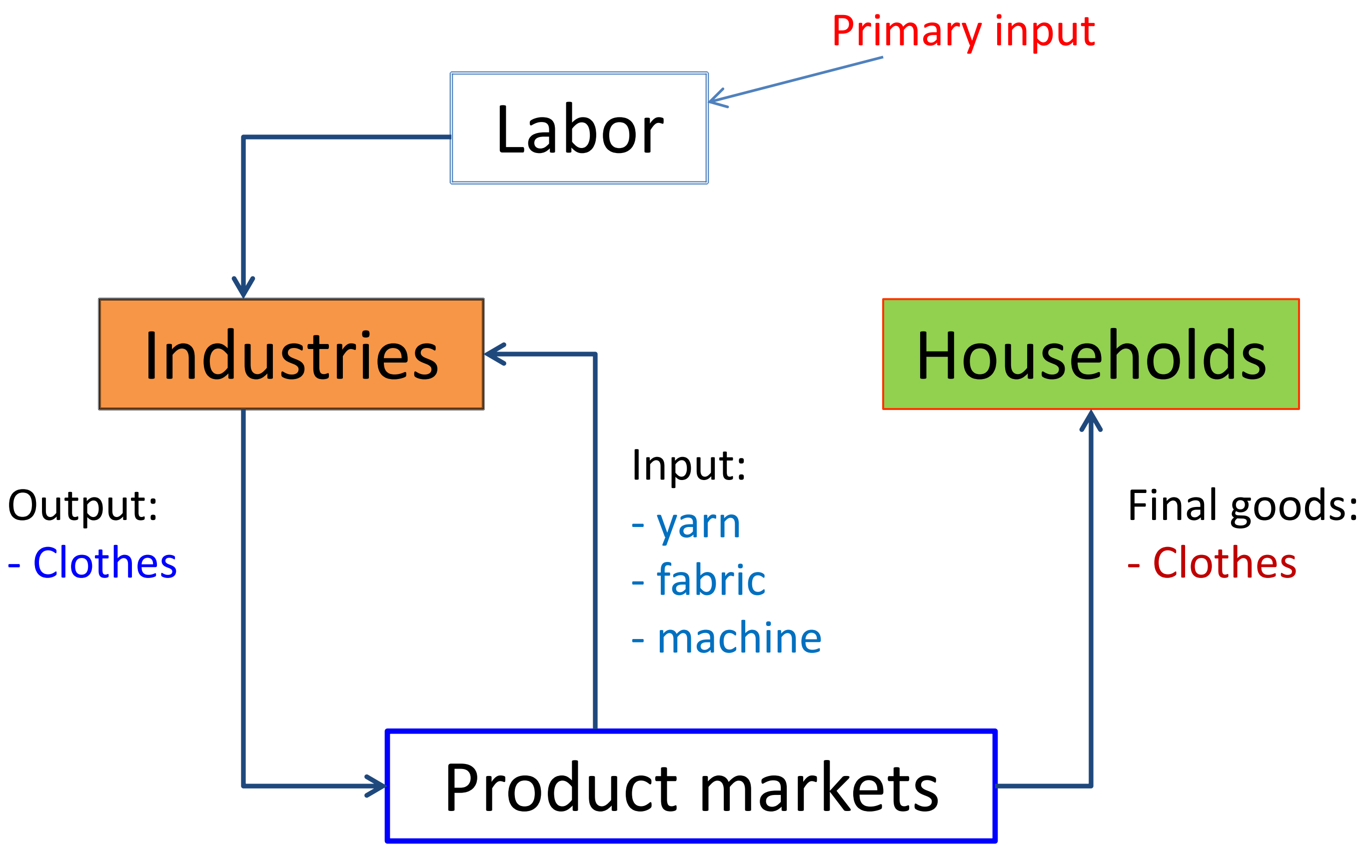
$$\boxed{Y^* = \frac{(a+I_0+G_0)h+iM_0}{ik+h[1-b(1-t)]}}$$

$$\boxed{r^* = \frac{(a+I_0+G_0)k-[1-b(1-t)M_0]}{ik+h[1-b(1-t)]}}$$

Leontif Input-Output Model (1)

- This model characterizes input-output relationships among *n* interlinked industries in an economy.
- It seeks to determine the output level that each of the *n* industries should produce to meet the total demand for that product.
- Applications: Use in production planning, e.g. planning for economic development of a country
- Assumptions:
  - Each industry produces only one homogenous commodity.
  - Each industry uses a fixed input ratio for the production of its output.
  - Production in every industry is subject to constant returns to scale.

Leontif Input-Output *Open* Model (2)



Leontif Input-Output Model (3)

- Consider an economy with *n* industries.
  - All the outputs are produced need to meet (i) the input demand of the same *n* industries and (ii) the final demand (e.g. consumer demand)
  - 2 types of inputs: (i) intermediate inputs and (ii) primary inputs (e.g. labor)
- Notations:
  - x_i = total number of units of good i that industry i is going to produce
  - a_{ij} = the number of units of good i needed to produce one unit of good j (a_{ij} = input coefficient)
  - $a_{ij}x_j$ = the number of units of good i needed to produce x_j unit of good j
  - d_i = final demand for good i

Leontif Input-Output Model (4)

- Equilibrium between supply and demand for each good *i*:

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + b_i$$

- The system of equations for *n* industries:

$$x_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1$$
$$x_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_n$$

- Input-coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Leontif Input-Output Model (4)

- Rearranging the above system of equations:

$$
\begin{aligned}
(1-a_{11})x_1 - \quad a_{12}x_2 - \dots - \quad a_{1n}x_n &= b_1 \\
-a_{21}x_1 \quad + (1-a_{22})x_2 - \dots - \quad a_{2n}x_n &= b_2 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\
-a_{n1}x_1 - \quad a_{n2}x_2 - \dots \quad + (1-a_{nn})x_n &= b_n
\end{aligned}
$$

- Leontif model in matrix form:

$$
\underbrace{\begin{bmatrix} (1-a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (1-a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (1-a_{nn}) \end{bmatrix}}_{\text{Leontif matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \quad \text{or} \quad (I-A)x = d
$$

- Given that $(I-A)^{-1}$ exists, the unique solution of the system is:

$$x^* = (I-A)^{-1}d$$

Leontif Input-Output Model (5)

- A numerical example:

Industry	Input Demand			Final Demand ($)
	Agriculture x_1	Manufacture x_2	Service x_3	
Agriculture	0.2	0.3	0.2	10
Manufacture	0.4	0.1	0.2	5
Service	0.1	0.3	0.2	6

- Questions:
  - What level of output should each industry produce?
  - Find the input coefficients of the primary input (labor) for each output, and determine the required amount of the primary input needed in this economy.

Leontif Input-Output Model (6)

- Write the input-coefficient matrix:

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.1 \end{bmatrix}$$

- The input-output system in the matrix form *(I-A)x = d*:

$$\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$$

Leontif Input-Output Model (7)

- Given that $\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix}^{-1} = \dfrac{1}{0.38}\begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix}$

- The solutions for the output levels to be produced are:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_2^* \end{bmatrix} = \frac{1}{0.384}\begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 24.84 \\ 20.68 \\ 18.36 \end{bmatrix}$$

- The level of primary input (labor) required for this economy:

$a_{01} = 0.3$; $a_{02} = 0.3$; $a_{01} = 0.4$

➔ $a_0 = 0.3(24.84) + 0.3(20.68) + 0.4(18.36) = 21$