Solution Part II: Exercise for Assignment 5

1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define relations
   $f : X \to Y$ by $f = \{(1, a), (2, a), (3, c)\}$,
   $g : X \to Y$ by $g = \{(1, a), (3, c)\}$, and
   $h : X \to Y$ by $h = \{(1, a), (2, a), (3, b), (3, c)\}$.

   (a) Draw the arrow diagrams of f, g, and h.

   (b) Show that f is a function, but g and h are not functions.

   (c) Find the domain of f, co-domain of f,and range of f.

   (d) What is the inverse image of a for the function f?

   (e) What is $f(3)$?

   Solution:

   (a) Draw the arrow diagrams of f, g, and h.

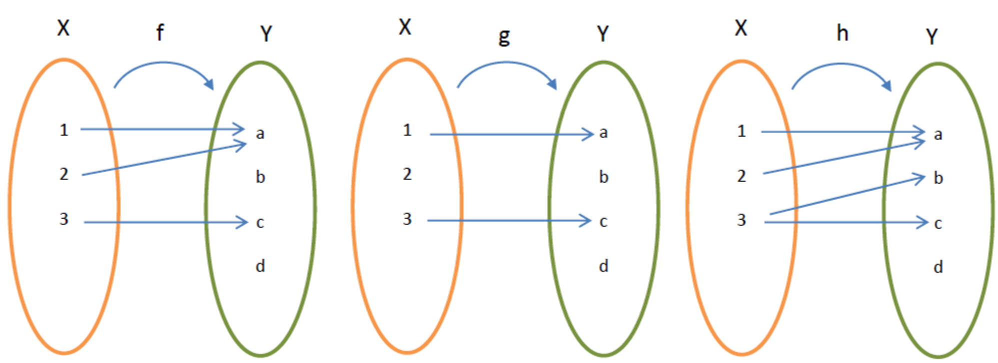

Figure 1: Problem 1(a)

   (b) Show that f is a function, but g and h are not functions.
   f is a function because(i) every element in the domain is used and (ii) each $x \in X$ gets mapped exactly once(for each $x \in X$, there is a unique $y \in Y$ so that $(x, y) \in f$, or if $(x, y_1), (x, y_2) \in f$ then $y_1 = y_2$).

   (c) Find the domain of f, co-domain of f,and range of f.
   The domain is $X = \{1, 2, 3\}$
   The co-domain is $Y = \{a, b, c, d\}$.
   The range is $\Big\{a, c\Big\}$.

   (d) What is the inverse image of a for the function f?
   The inverse image of a are the elements in X that get mapped to a:

$$\text{Inverse Image of } a = \{1, 2\}.$$

(e) What is $f(3)$?
Since $(3, c) \in f$, then $f(3) = c$.

2. Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For all $(x, y) \in A \times B$,

$$x \; R \; y \Leftrightarrow \; |x| = |y|.$$

and

$$x \; S \; y \;\; \Leftrightarrow \;\; x - y \text{ is even.}$$

State explicitly the sets $A \times B$, R, S, $R \cup S$, and $R \cap S$.

Solution:

- $A \times B = \{(x, y)|x \in A, y \in B\} = \{(-1, 1), (1, 1), (2, 1), (4, 1), (-1, 2), (1, 2), (2, 2), (4, 2)\}$
- $R = \{(x, y) \in A \times B||x| = |y|\} = \{(-1, 1), (1, 1), (2, 2)\}$
- $R = \{(x, y) \in A \times B|x - y \text{ is even}\} = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$
- $R \cup S = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$
- $R \cap S = \{(-1, 1), (1, 1), (2, 2)\}$

3. Let $A = \{1, 2, 3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A, and

$$X = \{x \in \mathcal{P}(A)| \; x \cap \{1\} \neq \emptyset\}.$$

Define a relation r from X to $\mathbb{Z}$ as

$$r = \{(x, y) \in X \times \mathbb{Z}| \; y = \text{ the number of elements in } x\}.$$

(a) List all elements in X.
(b) Draw an arrow diagram of r .
(c) Is r a function? If so, find the domain, co-domain, and range of r.

Solution:

(a) List all elements in X.
Since $\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$,

$$X = \{x \in \mathcal{P}(A)| \; x \cap \{1\} \neq \emptyset\} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}.$$

(b) Draw an arrow diagram of r .

(c) Is r a function? If so, find the domain, co-domain, and range of r.
Yes, r is a function because(i) every element in the domain is used and (ii) for each $x \in X$, there is a unique $y \in \mathbb{Z}$ so that $(x, y) \in r$, or if $(x, y_1), (x, y_2) \in r$ then $y_1 = y_2$ (i.e. each $x \in X$ gets mapped exactly once).
The domain is $X = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$
The co-domain is $\mathbb{Z} = \Big\{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \Big\}$.
The range is $\Big\{1, 2, 3\Big\}$.

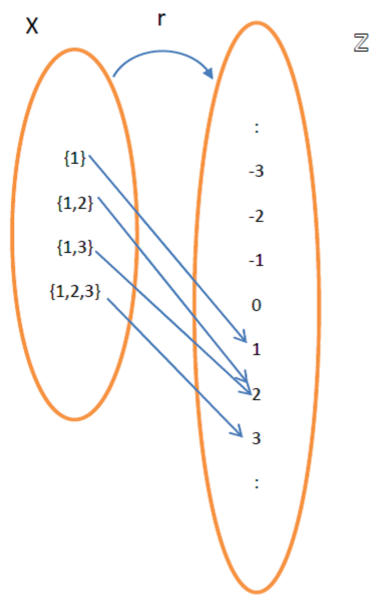

Figure 2: Problem 3(b)

4. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (x + 1, 2y - 3) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Yes, H is one-to-one. To prove this, we will show that for (x_1, y_1) and (x_2, y_2) in the domain, if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$. Suppose $H(x_1, y_1) = H(x_2, y_2)$, Then

$$\begin{aligned} H(x_1, y_1) &= H(x_2, y_2) \\ (x_1 + 1, 2y_1 - 3) &= (x_1 + 1, 2y_1 - 3) \end{aligned}$$

and this is equivalent to $x_1 + 1 = x_2 + 1 \quad$ and $\quad 2y_1 - 3 = 2y_2 - 3$, or

$$x_1 + 1 = x_2 + 1 \quad \Leftrightarrow \quad x_1 = x_2$$

$$2y_1 - 3 = 2y_2 - 3 \quad \Leftrightarrow \quad 2y_1 = 2y_2 \quad \Leftrightarrow \quad y_1 = y_2$$

i.e., $(x_1, y_1) = (x_2, y_2)$. That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$. ■

(b) Yes, H is onto. To prove this, we will show that for any (u, v) in the co-domain $\mathbb{R} \times \mathbb{R}$, there exists (x, y) from the domain $\mathbb{R} \times \mathbb{R}$ such that $H(x, y) = (u, v)$.
Suppose, temporarily that, there is (x, y) such that $H(x, y) = (u, v)$. Then,

$$\begin{aligned} H(x, y) &= (u, v) \\ (x + 1, 2y - 3) &= (u, v) \end{aligned}$$

i.e., we must have $x + 1 = u \quad \Leftrightarrow \quad \boxed{x = u - 1}$ and $2y - 3 = v \quad \Leftrightarrow \quad \boxed{y = \frac{v+3}{2}}$.
Since $u, v \in \mathbb{R}$, then $x = u - 1, y = \frac{v+3}{2} \in \mathbb{R}$.
That is, for *any* given (u, v) in the co-domain $\mathbb{R}\times\mathbb{R}$, we can find $(x, y) = \left(u - 1, \frac{v+3}{2}\right) \in \mathbb{R}\times\mathbb{R}$ in the domain, such that

$$H(x, y) = H\left(u - 1, \frac{v + 3}{2}\right) = H\left(u - 1 + 1, 2(\frac{v + 3}{2}) - 3\right) = (u, v + 3 - 3) = (u, v),$$

and hence H is onto. ■

(c) From (a) and (b), since H is both one-to-one and onto, then H bijective.
To find the inverse function, H^{-1}, of H, we recall from the definition

$$H^{-1}(u, v) = (x, y) \qquad \Leftrightarrow \qquad H(x, y) = (u, y).$$

Since we have from (b) that

$$H\left(u - 1, \frac{v + 3}{2}\right) = (u, v),$$

and hence the inverse function $H^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is given by

$$H^{-1}(u, v) = \left(u - 1, \frac{v + 3}{2}\right).$$

■

5. Define functions $f_1 : [0,2) \to \mathbb{R}$ as

$$f_1(x) = x^2$$

and define $f_2 : [2,\infty) \to \mathbb{R}$ as

$$f_2(x) = 3x - 2.$$

Let $F : [0,\infty) \to [0,\infty)$ be a function defined by using f_1 and f_2:
$F(x) = f_1(x)$, for $x \in [0,2)$, and $F(x) = f_2(x)$, for $x \in [2,\infty)$. That is,

$$F(x) = \begin{cases} x^2, & x \in [0,2) \\ 3x-2, & x \in [2,\infty). \end{cases}$$

(a) Find the domain, co-domain, and range for each of the functions f_1, f_2, and F.

(b) Construct the composite functions $f_1 \circ f_2$, $f_2 \circ f_1$, and $f_1 \circ F$ (if possible). Determine the domains and ranges for these composite functions.

(c) Are f_1 and f_2 injective? Explain.

(d) Is the function F bijective? If so, find the **inverse function** of F.

Answer

(a) The domain, co-domain, and range for each of the functions f_1, f_2, and F.
The domain of f_1 is $D_{f_1} = [0,2)$, and the range is $R_{f_1} = [0,4)$, which comes from:

$$0 \le x < 2 \quad \Rightarrow \quad 0 \le x^2 < 4 \quad \Rightarrow \quad 0 \le f_1(x) < 4.$$

The domain of f_2 is $D_{f_2} = [2,\infty)$, and the range is $R_{f_2} = [4,\infty)$, which comes from:

$$2 \le x \quad \Rightarrow \quad 3x - 2 \ge 4 \quad \Rightarrow \quad f(x_2) \ge 4.$$

The domain of F is $D_F = (\infty,2) \cup (2,\infty) = [0,\infty)$, and the range is $R_F = [0,\infty)$, which comes from:

$$\text{For } x =\in [0,2), 0 \le x < 2 \quad \Rightarrow \quad 0 \le x^2 < 4 \quad \Rightarrow \quad F(x) \in [0,4).$$

$$\text{For } x =\in [2,\infty) \quad \Rightarrow \quad 2 \le x \quad \Rightarrow \quad 3x-2 \ge 4 \quad \Rightarrow \quad F(x) \in [4,\infty).$$

(b) Construct the composite functions $f_1 \circ f_2$, $f_2 \circ f_1$, and $f_1 \circ F$ (if possible). Determine the domains and ranges for these composite functions.

$f_1 \circ f_2$ can be defined only when $D_{f_1} \cap R_{f_2}$.
From the previous part, we have $D_{f_1} \cap R_{f_2} = [0,2) \cap [4,\infty) = \emptyset$, so we cannot construct $f_1 \circ f_2$.

■

$f_2 \circ f_1$ can be defined only when $D_{f_2} \cap R_{f_1}$.
From the previous part, we have $D_{f_2} \cap R_{f_1} = [2,\infty) \cap [0,4) \neq \emptyset$, so we can construct $f_2 \circ f_1$: for $x \in D_{f_1}$,

$$(f_2 \circ f_1)(x) = f_2(f_1(x)) = f_2(x^2) = 3x^2 - 2.$$

For the domain $D_{f_2\circ f_1}$ of $f_2\circ f_1$, we must have $D_{f_2\circ f_1}\subset D_{f_1}$ and also for $x\in D_{f_1}$, to have $f_2(f_1(x))$ well-defined, we must have $f_1(x)\in D_{f_2}$:

$$f_1(x)\geq 2 \quad\Rightarrow\quad x^2\geq 2 \quad\Rightarrow\quad x\geq\sqrt{2} \text{ or } x\leq-\sqrt{2}.$$

So the domain $D_{f_2\circ f_1}$ is $\{(-\infty,-\sqrt{2}]\cup[\sqrt{2},\infty)\}\cap D_{f_1}=\{(-\infty,-\sqrt{2}]\cup[\sqrt{2},\infty)\}\cap[0,2]=[\sqrt{2},2]$. From $(f_2\circ f_1)(x)=3x^2-2$, for $x\in[\sqrt{2},2]$,

$$\sqrt{2}\leq x\leq 2 \quad\Rightarrow\quad 3\cdot 2-2\leq 3x^2-2\leq 3\cdot 4-2 \quad\Rightarrow\quad 4\leq 3x^2-2\leq 10 \quad\Rightarrow\quad (f_2\circ f_1)(x)\in[4,10]$$

and the range of $f_2\circ f_1$ is $[4,10]$. ■

$f_1\circ F$ can be defined only when $D_{f_1}\cap R_F$.
we have $D_{f_1}\cap R_F=[0,2)\cap[0,\infty)\neq\emptyset$, so we can construct $f_1\circ F$: for $x\in D_F$,

$$(f_1\circ F)(x)=f_1(F(x))$$

and we have to choose the formula for $F(x)$ such that

$$F(x)\in D_{f_1} \quad\Leftrightarrow\quad 0\leq F(x)<2 \quad\Leftrightarrow\quad F(x)=x^2.$$

Note that if we use $F(x)=3x-2$, we will have $F(x)\geq 4$ because $x\geq 2$. Hence,

$$(f_1\circ F)(x)=f_1(x^2)=x^4.$$

The domain can be obtained from $x\in D_F$,

$$F(x)\in D_{f_1} \quad\Leftrightarrow\quad 0\leq x^2<2 \quad\Leftrightarrow\quad x^2-2<0 \quad\Leftrightarrow\quad (x-\sqrt{2})(x+\sqrt{2})<0 \quad\Leftrightarrow\quad -\sqrt{2}<x<\sqrt{2}$$

which implies that the domain $D_{f_1\circ F}$ is $[0,\infty]\cap(-\sqrt{2},\sqrt{2})=[0,\sqrt{2})$. From $(f_1\circ F)(x)=x^4$, for $x\in[0,\sqrt{2})$,

$$0\leq x<\sqrt{2} \quad\Rightarrow\quad 0\leq x^4<4$$

and the range of $f_1\circ F$ is $R_{f_1\circ F}=[0,4)$.

(c) Are f_1 and f_2 injective? Explain.
Yes, both f_1 and f_2 are injective. It can be shown that f_1 is injective as follows.
Let $x_1,x_2\in D_{f_1}=[0,2)$. We want to show that

$$f_1(x_1)=f_1(x_2) \quad\text{implies}\quad x_1=x_2.$$

Suppose $f_1(x_1)=f_1(x_2)$. Then, for $x_1,x_2\in[0,2)$,

$$\begin{aligned} f_1(x_1) &= f_1(x_2)\\ x_1^2 &= x_2^2\\ x_1 &= x_2. \end{aligned}$$

Note that we cannot have $x_1=-x_2$ since $x_1,x_2\geq 0$. Hence, f_1 is injective. ■

It can be shown that f_2 is injective as follows.
Let $x_1,x_2\in D_{f_2}=[2,\infty)$. We want to show that

$$f_2(x_1)=f_2(x_2) \quad\text{implies}\quad x_1=x_2.$$

Suppose $f_2(x_1) = f_2(x_2)$. Then, for $x_1, x_2 \geq 0$,

$$\begin{aligned} f_2(x_1) &= f_2(x_2) \\ 3x_1 - 2 &= 3x_2 - 2 \\ x_1 &= x_2, \end{aligned}$$

by adding 2 and dividing by 3 throughout the second equality. Hence, f_2 is injective. ■

(d) Is the function F bijective? If so, find the **inverse function** of F.

Yes, F is bijective. This can be verified by showing that F is one-to-one and onto.

F is one-to-one.

This can be shown as follows.

By definition, for $x_1, x_2 \in D_F = [0, \infty)$, F is one-to-one if

$$F(x_1) = F(x_2) \quad \text{implies} \quad x_1 = x_2.$$

Since there are two formulas for F depending on the value of x, we will consider 3 cases for the values of x_1 and x_2: (i) $x_1, x_2 \in [0, 2)$, (ii) $x_1, x_2 \in [2, \infty)$, (iii) $x_1 \in [0, 2)$ $x_2 \in [2, \infty)$.

(i) For $x_1, x_2 \in [0, 2)$, we will use $F(x) = x^2$ and, just like f_1, we have

$$F(x_1) = F(x_2) \quad \Rightarrow \quad x_1^2 = x_2^2 \quad \Rightarrow \quad x_1 = x_2.$$

(ii) For $x_1, x_2 \in [2, \infty)$, we will use $F(x) = 3x - 2$ and, just like f_2, we have

$$F(x_1) = F(x_2) \quad \Rightarrow \quad 3x_1 - 2 = 3x_2 - 2 \quad \Rightarrow \quad x_1 = x_2.$$

(iii) For $x_1 \in [0, 2)$ and $x_2 \in [2, \infty)$, we will use the contrapositive of the definition:

$$x_1 \neq x_2 \quad \Rightarrow \quad F(x_1) \neq F(x_2).$$

In this case, since x_1 and x_2 are from a disjoint intervals, then $x_1 \neq x_2$. We see that,

$$x_1 \in [0, 2) \quad \Rightarrow \quad F(x_1) = x_1^2 \in [0, 4)$$

as also shown in part (a) for range of F. and

$$x_2 \in [2, \infty) \quad \Rightarrow \quad F(x_2) = 3x_2 - 2 \in [4, \infty).$$

That is, $F(x_1) \neq F(x_2)$ because they are in two disjoint intervals $[0, 4)$ and $[4, \infty)$.

From cases (i)-(iii), we can conclude that F is injective.

F is onto.

Since F has two formulas and they give function values in two different disjoint intervals: $[0, 4)$ and $[4, \infty)$, as shown above and as also shown in (a), we will verify that F is onto using two cases.

Let $y \in [0, \infty)$, which is the co-domain of F.

- If $y \in [0,4)$, we can find $x \in D_F = [0,\infty)$ as follows.
  Set $y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$. So,

$$F(x) = F(\sqrt{y}) = (\sqrt{y})^2 = y.$$

  Since $0 \le y < 4$, then $0 \le \sqrt{y} < 2$, and $x = \sqrt{y} \in [0,2) \subset D_f$.
- If $y \in [4,\infty)$, we can find $x \in D_F = [0,\infty)$ as follows.
  Set $y = 3x - 2 \quad \Rightarrow \quad x = \frac{y+2}{3}$. So

$$F(x) = F\left(\frac{y+2}{3}\right) = 3\left(\frac{y+2}{3}\right) - 2 = y.$$

  Since $y \in [4,\infty)$ or $y \ge 4$, then $\frac{y+2}{3} \ge \frac{4+2}{3} = 2$, and $x = \frac{y+2}{3} \in [2,\infty) \subset D_F$.

The above two cases cover all possible values of y in the co-domain $[0,\infty)$ of F. That is, for any given $y \in [0,\infty)$, we can find $x \in D_F$ such that $F(x) = y$ and hence, F is onto.
Since F is one-to-one and onto, then it is bijective. ■
We now will find the inverse, F^{-1}, of F, which satisfies

$$F(x) = y \quad \Leftrightarrow \quad F^{-1}(y) = x.$$

When we showed that F is onto, we have

- For $y \in [0,4)$, there is $x = \sqrt{y} \in [0,2)$ such that

$$F(\sqrt{y}) = y.$$

- For $y \in [4,\infty)$, there is $x = \frac{y+2}{3} \in [2,\infty)$ such that

$$F\left(\frac{y+2}{3}\right) = y.$$

That is, we can defined $F^{-1} : [0,\infty) \to [0,\infty)$ as

$$F^{-1}(y) = \begin{cases} \sqrt{y}, & x \in [0,4) \\ \frac{y+2}{3}, & y \in [4,\infty). \end{cases}$$

■

6. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$

   (b) $f(x) = x^5, \quad g(x) = x^{1/5}.$

   Solution: First notice that all domains and ranges of f and g for both (a) and (b) are the set of real numbers $\mathbb{R}$ and therefore it is possible to define $f \circ g$ and $g \circ f$.

(a) $f(x) = \frac{x}{\sqrt{x^2+1}}$, $g(x) = x^3 + 1$. The function $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f \circ g)(x) = f(g(x)) = \frac{x^3+1}{\sqrt{(x^3+1)^2+1}}$$

and function $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by

$$(g \circ f)(x) = g(f(x)) = \left(\frac{x}{\sqrt{x^2+1}}\right)^3 + 1.$$

So we have $(f \circ g)(2) = \frac{2^3+1}{\sqrt{(2^3+1)^2+1}} = \frac{9}{\sqrt{82}}$ and $(g \circ f)(2) = \left(\frac{2}{\sqrt{2^2+1}}\right)^3 + 1 = \frac{8+5\sqrt{5}}{5\sqrt{5}}$. Notice that since $(f \circ g)(2) \neq (g \circ f)(2)$, then $f \circ g \neq g \circ f$. ■

(b) $f(x) = x^5$, $g(x) = x^{1/5}$. The function $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f \circ g)(x) = f(g(x)) = \left(x^{1/5}\right)^5 = x.$$

and function $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by

$$(g \circ f)(x) = g(f(x)) = \left(x^5\right)^{1/5} = x.$$

Notice that since $(f \circ g)(x) = (g \circ f)(x)$ for all $x \in \mathbb{R}$, then $f \circ g = g \circ f$. So we have $(f \circ g)(2) = (g \circ f)(2) = 2$ ■

7. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.

(a) Compute $f \circ f$ and determine its domain.

(b) Determine whether f is bijective. If so, find the inverse function f^{-1} and $f \circ f^{-1}$.

Solution:

(a) The composite function $f \circ f$ can be defined when $D_f \cap R_f \neq \emptyset$. The domain of f is $D_f = \mathbb{R} - \{1\}$. To find the range of f, we have to find all possible values of $y = f(x)$, which can be found by first set $y = f(x) = \frac{2x+1}{1-x}$ and then solve y in term of x:

$$y = \frac{2x+1}{1-x} \quad \Leftrightarrow \quad y - yx = 2x + 1 \quad \Leftrightarrow \quad yx + 2x = y - 1 \quad \Leftrightarrow \quad x = \frac{y-1}{y+2}$$

which implies that y can be any real number except -2 and the range is $R_f = \mathbb{R} - \{-2\}$. Since $D_f \cap R_f = (\mathbb{R} - \{1\}) \cap (\mathbb{R} - \{-2\}) \neq \emptyset$, we can define $f \circ f$ as:

$$(f \circ f)(x) = f\left(\frac{2x+1}{1-x}\right) = \frac{2\frac{2x+1}{1-x} + 1}{1 - \frac{2x+1}{1-x}} = \frac{\frac{4x+2+1-x}{1-x}}{\frac{1-x-2x-1}{1-x}} = \frac{3x+3}{-3x} = -\frac{x+1}{x}.$$

The domain of $D_{f \circ f}$ can be obtained from $x \in D_f = \mathbb{R} - \{1\}$ and using the fact that: we can evaluate $f(f(x))$, only when

$$f(x) \in D_f \quad \Rightarrow \quad f(x) \neq 1 \quad \text{or} \quad \frac{2x+1}{1-x} \neq -1 \quad \Leftrightarrow \quad 1-x = 2x+1 \quad \Leftrightarrow \quad 3x \neq 0 \quad \Leftrightarrow \quad x \neq 0.$$

That is, the domain of $f \circ f$ is $D_f \cap (\mathbb{R} - \{0\}) = \mathbb{R} - \{0, 1\}$ ■

(b) The function f is bijective.

f **is one-to-one**. Let $x_1, x_2 \in D_f = \mathbb{R} - \{1\}$. We want to show that

$$f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2.$$

Suppose $f(x_1) = f(x_2)$. Then, for $x_1, x_2 \in \mathbb{R} - \{1\}$,

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{2x_1+1}{1-x_1} &= \frac{2x_2+1}{1-x_2} \\ (2x_1+1)(1-x_2) &= (2x_2+1)(1-x_1) \\ 2x_1+1-2x_1x_2-x_2 &= 2x_2+1-2x_2x_1-x_1 \\ 2x_1-x_2 &= 2x_2-x_1 \\ 3x_1 &= 3x_2 \\ x_1 &= x_2. \end{aligned}$$

That is, f is one-to-one. ■

f **is onto**. For a given $y \in \mathbb{R} - \{-2\}$, which is the co-domain, we can show that f is onto by first setting $y = f(x)$:

$$y = \frac{2x+1}{1-x} \quad \Leftrightarrow \quad y - yx = 2x+1 \quad \Leftrightarrow \quad yx + 2x = y - 1 \quad \Leftrightarrow \quad x = \frac{y-1}{y+2}.$$

Since $y \in \mathbb{R} - \{2\}$, then $y \neq -2$ and x is well-defined real number in $\mathbb{R}$. To show that $f \in \mathbb{R} - \{1\}$, we will try to set $1 = x = \frac{y-1}{y+2}$ and we will see that

$$y - 1 = y + 2 \quad \Rightarrow \quad -1 = 2,$$

which is impossible and so $x = \frac{y-1}{y+2} \neq 1$. That is, given $y \in \mathbb{R} - \{-2\}$, we can find $x = \frac{y-1}{y+2} \in D_f = \mathbb{R} - \{1\}$ such that

$$f(x) = f\left(\frac{y-1}{y+2}\right) = \frac{2\left(\frac{y-1}{y+2}\right)+1}{1-\left(\frac{y-1}{y+2}\right)} = \frac{\left(\frac{2y-2+y+2}{y+2}\right)}{\left(\frac{y+2-y+1}{y+2}\right)} = \frac{\left(\frac{3y}{y+2}\right)}{\left(\frac{3}{y+2}\right)} = y \tag{1}$$

and, hence, f is onto. ■

To find the inverse, we can set $y = f(x) = \frac{2x+1}{1-x}$ as we did earlier and

$$y = \frac{2x+1}{1-x} \quad \Leftrightarrow \quad y - yx = 2x+1 \quad \Leftrightarrow \quad yx + 2x = y - 1 \quad \Leftrightarrow \quad x = \frac{y-1}{y+2},$$

which implies that the inverse function $f^{-1} : \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$ is given by

$$f^{-1}(y) = \frac{y-1}{y+2}$$

or, equivalently, $f^{-1}(x) = \frac{x-1}{x+2}$ (by changing the notation of the variable from y to x). Note that we can use the definition by using (1): $f\left(\frac{y-1}{y+2}\right) = y \quad \Leftrightarrow \quad f^{-1}(y) = \frac{y-1}{y+2}$. ■

$\underline{f \circ f^{-1}}$
Note that $D_f = R_{f^{-1}} = \mathbb{R} - \{1\}$ and $R_f = D_{f^{-1}} = \mathbb{R} - \{-2\}$. So $D_f \cap R_{f^{-1}} = \mathbb{R} - \{1\} \neq \emptyset$ and we can define $f \circ f^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ by

$$(f \circ f^{-1})(x) = f\left(\frac{x-1}{x+2}\right) = \frac{2\left(\frac{x-1}{x+2}\right) + 1}{1 - \left(\frac{x-1}{x+2}\right)} = \frac{\left(\frac{2x-2+x+2}{x+2}\right)}{\left(\frac{x+2-x+1}{x+2}\right)} = \frac{\left(\frac{3x}{x+2}\right)}{\left(\frac{3}{x+2}\right)} = x.$$

That is, $\boxed{(f \circ f^{-1})(x) = x}$ is an identity matrix on $\mathbb{R} - \{1\}$. Note that the above process is similar to (1). ■