Question 1:

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Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?

1.2 Is the production function increasing with respect to K and L?

1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.

1.4 Use the Hessian matrix. Proof that the production function is concave.

1.1) $Q = f(K,L) = 18 \times [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$

$K_0 L_0$; $Q_0 = 18 \times [0.2K_0^{-0.4} + 0.8L_0^{-0.4}]^{-2.5}$

tK_0, tL_0 ; $Q = 18 \times [0.2(tK_0)^{-0.4} + 0.8(tL_0)^{-0.4}]^{-2.5}$

$Q = (t^{-0.4})^{-2.5} \times 18 \times [(0.2K_0^{-4} + 0.8L_0^{-0.4}]^{-2.5}$

$Q = t^1 \times 18 \times [(0.2K_0^{-4} + 0.8L_0^{-0.4}]^{-2.5}$

$\therefore Q = t^1 \times Q_0$

So, this is constant return to scale because it has degree of Hm equal to 1.

1.2) $\frac{\partial Q}{\partial K} = 18 \times (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5-1} \times (0.2)(-0.4)K^{-0.4-1}$

$= 18 \times (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \times -0.08K^{-1.4}$

$= 18 \times (0.2K^{-1.4})[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}$

$\frac{\partial Q}{\partial L} = 18 \times (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \times (0.8)(-0.4)L^{-0.4-1}$

$= 18 \times (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \times -0.32L^{-1.4}$

$= 18 \times (0.8L^{-1.4})[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}$

Both of them are greater than 0, so the production function increasing with respect to K and L.

1.3) $f(K,L) = 0$

$$f(K,L) = 18 \times \left[0.2K^{-0.4} + 0.8L^{-0.4}\right]^{-2.5} = 0$$

$$MRTS = \frac{dL}{dK} = -\frac{F_K}{F_L} = \frac{\cancel{18} \times (0.8L^{-1.4})\cancel{\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}}}{\cancel{18} \times (0.2K^{-1.4})\cancel{\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}}}$$

$$= \frac{0.8L^{-1.4}}{0.2K^{-1.4}}$$

$$= \frac{4K^{1.4}}{L^{1.4}} \quad \text{Ans}$$

1.4) $\nabla u = \begin{bmatrix} u_K \\ u_L \end{bmatrix} \longrightarrow H = \begin{bmatrix} u_{KK} & u_{KL} \\ u_{LK} & u_{LL} \end{bmatrix}$

$$u_K = \frac{\partial u}{\partial K} = (3.6K^{-1.4})\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}, \quad u_L = \frac{\partial u}{\partial L} = (14.4L^{-1.4})\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}$$

$$u_{KK} = \frac{\partial u_K}{\partial K} = (3.6K^{-1.4})(-3.5)\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5}(-0.08K^{-1.4}) + \left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}(-5.04K^{-2.4})$$

$$= (1.008K^{-2.8})\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5} + \left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}(-5.04K^{-2.4})$$

$$u_{KL} = \frac{\partial u_K}{\partial L} = (3.6K^{-1.4})(-3.5)\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5}(-0.32L^{-1.4}) + \left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}(0)$$

$$= (4.032K^{-1.4}L^{-1.4})\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5}$$

$$u_{LK} = \frac{\partial u_L}{\partial K} = (14.4L^{-1.4})(-3.5)\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5}(-0.08K^{-1.4}) + \left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}(0)$$

$$= (4.032K^{-1.4}L^{-1.4})\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5}$$

$$u_{LL} = \frac{\partial u_L}{\partial L} = (14.4L^{-1.4})(-3.5)\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5}(-0.32L^{-1.4}) + \left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}(-20.16L^{-2.4})$$

$$= (16.128L^{-2.8})\left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-4.5} + \left[0.2K^{-0.4}+0.8L^{-0.4}\right]^{-3.5}(-20.16L^{-2.4})$$

$$|H_1| = \left|(1.008K^{-2.8})[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4}+0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})\right|$$
$$= (1.008K^{-2.8})[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4}+0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})$$

$$|H_2| = \Big[\{(1.008K^{-2.8})[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4}+0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})\}$$
$$\times \{(16.128L^{-2.8})[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4}+0.8L^{-0.4}]^{-3.5}(-20.16L^{-2.4})\}\Big]$$
$$-\Big[\{(4.032K^{-1.4}L^{-1.4})[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5}\} \times \{(4.032K^{-1.4}L^{-1.4})[0.2K^{-0.4}+0.8L^{-0.4}]^{-4.5}\}\Big]$$

$|H_1| < 0$; $\forall K, \forall L$

$|H_2| > 0$; $\forall K, \forall L$

$\therefore$ H is negative definite ; $d^2Q < 0$

So, production function is concave.

Question 2: Define f(x,y) for all (x,y) by

$$f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

2.1 Derive the Hessian matrix of $f(x,y)$.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} e^{x+y}+e^{x-y} & e^{x+y}-e^{x-y} \\ e^{x+y}-e^{x-y} & e^{x+y}+e^{x-y} \end{bmatrix}$$

2.2 Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).

$$|H_1| = |e^{x+y}+e^{x-y}| \ ; \ \forall x, \forall y \rightarrow |H_1| > 0$$

$$|H_2| = \begin{vmatrix} e^{x+y}+e^{x-y} & e^{x+y}-e^{x-y} \\ e^{x+y}-e^{x-y} & e^{x+y}+e^{x-y} \end{vmatrix}$$

$$= (e^{x+y}+e^{x-y})(e^{x+y}+e^{x-y}) - (e^{x+y}-e^{x-y})(e^{x+y}-e^{x-y}) > 0$$

$d^2f > 0$ for $\forall x$, $\forall y$

so $f(x,y)$ is globally convex

2.3 Find the *global* extrema of f(x,y). What type of extrema is it?

$$\frac{\partial f}{\partial x} = e^{x+y}+e^{x-y}-\frac{3}{2} = 0$$

$$e^{x+y}+e^{x-y} = \frac{3}{2} \quad - (1)$$

$$\frac{\partial f}{\partial y} = e^{x+y}-e^{x-y}-\frac{1}{2} = 0$$

$$e^{x+y}-e^{x-y} = \frac{1}{2} \quad - (2)$$

(1)+(2) ; $e^{x+y}+e^{x-y}+e^{x+y}-e^{x-y} = 2$

$$2e^{x+y} = 2$$

$$e^{x+y} = 1$$

$$e^{x+y} = e^0$$

$$x+y = 0$$

$$x = -y$$

plug x in (1) : $e^{-y+y}+e^{-y-y} = \frac{3}{2}$

$$1 + e^{-2y} = \frac{3}{2}$$

$$e^{-2y} = \frac{1}{2}$$

$$-2y = \ln\frac{1}{2}$$

$$-2y = \ln 1 - \ln 2$$

$$-2y = -\ln 2$$

$$y = \frac{\ln 2}{2}$$

$$x = -\frac{\ln 2}{2}$$

$\therefore$ critical point x,y is $\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$

At critical point, $\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is the local minimizer because $f(x,y)$ is globally convex, $\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is global minimizer.

The global minimum is :

$$f\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right) = e^{-\frac{\ln 2}{2}+\frac{\ln 2}{2}} + e^{-\frac{\ln 2}{2}-\frac{\ln 2}{2}} - \frac{3}{2}\left(-\frac{\ln 2}{2}\right) - \frac{1}{2}\left(\frac{\ln 2}{2}\right)$$

$$= e^0 + e^{-\ln 2} + \frac{3\ln 2}{4} - \frac{\ln 2}{4}$$

$$= 1 + \frac{1}{e^{\ln 2}} + \frac{\ln 2}{2}$$

$$= 1 + \frac{1}{2} + \frac{\ln 2}{2}$$

$$= \frac{3+\ln 2}{2}$$

Question 3:

A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

Suppose we increase L&K by t

Then,

$$(tK)^{\frac{1}{3}}(tL)^{\frac{2}{3}} = t^{\frac{1}{3}} \cdot t^{\frac{2}{3}} K^{\frac{1}{3}} L^{\frac{2}{3}}$$
$$= t K^{\frac{1}{3}} L^{\frac{2}{3}}$$
$$= tQ \#$$

$\therefore$ Degree of homogeneity is equal to 1, reflecting constant return to scale.

From now on, assume that c = ¼ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$TR = P \cdot Q = Q^{-c} Q = Q^{1-\frac{1}{4}} = \left(K^{\frac{1}{3}} L^{\frac{2}{3}}\right)^{\frac{3}{4}}$$

$$TC = rK + wL$$

$$\text{Profit Function} = TR - TC$$
$$= \left(K^{\frac{1}{3}} L^{\frac{2}{3}}\right)^{\frac{3}{4}} - (rK + wL)$$
$$= \left(K^{\frac{1}{3}} L^{\frac{2}{3}}\right)^{\frac{3}{4}} - rK - wL$$
$$= K^{\frac{1}{4}} L^{\frac{1}{2}} - rK - wL \#$$

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

$$\pi(K,L) = K^{\frac{1}{4}} L^{\frac{1}{2}} - rK - wL$$

F.O.C

$$\frac{\partial \pi}{\partial K} = \frac{1}{4} K^{-\frac{3}{4}} L^{\frac{1}{2}} - r = 0$$

$$4rK^{3/4} = L^{\frac{1}{2}}$$

$$K = \left(\frac{L^{\frac{1}{2}}}{4r}\right)^{4/3} \quad \text{— (1)}$$

$$\frac{\partial \pi}{\partial L} = \frac{1}{2} K^{\frac{1}{4}} L^{-\frac{1}{2}} - w = 0$$

$$2wL^{\frac{1}{2}} = K^{\frac{1}{4}}$$

$$L = \left(\frac{K^{\frac{1}{4}}}{2w}\right)^2 \quad \text{— (2)}$$

Substitute (2) into (1)

$$K = \left[\frac{\left(\left(\frac{K^{\frac{1}{4}}}{2w}\right)^2\right)^{\frac{1}{2}}}{4r}\right]^{4/3}$$

$$K = \left[\frac{\left(\frac{K^{\frac{1}{4}}}{2w}\right)}{4r}\right]^{4/3}$$

$$K = \frac{K^{1/3}}{(2w)^{4/3}} \cdot \frac{1}{(4r)^{4/3}}$$

$$(8wr)^{4/3} = \frac{K^{1/3}}{K}$$

$$(8wr)^{4/3} = K^{-2/3}$$

$$K^{2/3} = \frac{1}{(8wr)^{4/3}}$$

$$K^* = \left[\frac{1}{(8wr)^{4/3}}\right]^{3/2} = \frac{1}{(8wr)^2} \#$$

$$L^* = \left[\frac{\left(\frac{1}{(8wr)^2}\right)^{1/4}}{2w}\right]^2$$

$$= \frac{1}{8wr} \cdot \frac{1}{(2w)^2}$$

$$= \frac{1}{32w^3r} \#$$

The demand for factor inputs:

Capital Demand

$$P \cdot MP_K = VMP_K = \frac{1}{(8wr)^2} \#$$

Labor Demand

$$P \cdot MP_L = VMP_L = \frac{1}{32w^3r} \#$$

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

$$L^* = \frac{1}{32w^3r}$$

$\frac{\partial L^*}{\partial w} = -\frac{3}{32w^4r}$ # ; when the wage increases by one unit, the optimal amount of labor falls by $\left(\frac{3}{32w^4r}\right)$ unit.

$\frac{\partial L^*}{\partial r} = -\frac{1}{32w^3r^2}$ # ; when the unit price of capital rises by one unit, the optimal amount of labor falls by $\left(\frac{1}{32w^3r^2}\right)$ unit.

3.5) Confirm your answer with the second-order condition.

$$\pi_K = \frac{1}{4}K^{-\frac{3}{4}}L^{\frac{1}{2}} - r \; ; \; \pi_L = \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}} - w$$

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} \frac{-3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} & \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} \\ \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} & -\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}} \end{bmatrix}$$

$$|H_1| = \left| \frac{-3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} \right| = \frac{-3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}}$$

$$|H_2| = \begin{vmatrix} \frac{-3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} & \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} \\ \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} & -\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}} \end{vmatrix} = \left(\frac{-3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}}\right)\left(-\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}}\right) - \left(\frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}\right)\left(\frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}\right)$$

$$= \frac{3}{64}K^{-\frac{3}{2}}L^{-1} - \frac{1}{64}K^{-\frac{3}{2}}L^{-1} = \frac{1}{32}K^{-\frac{3}{2}}L^{-1}$$

Plug in $K^* = \frac{1}{(8wr)^2}$; $L^* = \frac{1}{32w^3r}$ into $|H_1|$, $|H_2|$

$|H_1| < 0$ at (K^*, L^*)
$|H_2| > 0$ at (K^*, L^*)

$\therefore$ H is negative definite at K^*, L^* ; $d^2\pi < 0$ at K^*, L^*

K^*, L^* is a local minimizer

Try any (K, L) other than (K^*, L^*) $\longrightarrow$ Always $|H_1| < 0$ and $|H_2| > 0$

$\therefore$ Always negative definite ; $d^2\pi$ is always less than 0.

π is globally concave local maximizer is also a global maximizer.