(a) $\int x e^{x} \, dx$ → look complicated because this can be viewed as $f(x) \cdot g(x)$

cannot use substitution because if you try to diff e^x w.r.t. $x \Rightarrow$ get e^x not x

or if try to diff x w.r.t. $x \Rightarrow$ get 1 not e^x

$\therefore$ for $\int f(x) \cdot g(x) \, dx$ that cannot be integrated using substitution $\Rightarrow$ Think of integration by part

Integration by part

$$\underbrace{\int_a^b f(x) g'(x) \, dx}_{\text{left han side (LHS)}} = \underbrace{\overbrace{\left[f(x) g(x) \right]_a^b}^{\text{RHS (1)}} - \overbrace{\int_a^b f'(x) g(x) \, dx}^{\text{RHS (2)}}}_{\text{right hand side (RHS)}}$$

The question is between x & e^x which one should be $f(x)$, which one should be $g'(x)$ in the integration by part???

If you look at the right hand side of integration by part formulae you will notice that you need to get $g(x)$ from $g'(x)$

$\therefore$ choose the function to be $g'(x)$ such that it can be integrated easily (1)

It'll be benefitial if the $g(x)$ obtained is not too complicated because you will need to integrate $\int f'(x) g(x) \, dx$ (see RHS (2) above). Finally try to choose $f(x)$ that results in $f'(x)$ that looks less complicated than $f(x)$.

$\therefore$ For this example choose $f(x) = x$ $\quad$ $g'(x) = e^x$

<u>Note</u> $g'(x) = e^x \Rightarrow g(x) = e^x + C_1$ (integrate easily & does not look more complicated than what it was)

$f(x) = x \Rightarrow f'(x) = 1$

(So RHS (2) will look not too hard to integrate $\Rightarrow \int e^x \, dx$

$$\therefore \int \overset{f(x)}{x}\,\overset{g'(x)}{e^x}\,dx = \Big[\overset{f(x)}{x}\,\overset{g(x)}{(e^x+c_1)}\Big] - \int \overset{f'(x)}{(1)}\,\overset{g(x)}{(e^x+c_1)}\,dx$$

$$= xe^x + \cancel{c_1 x} - e^x - \cancel{c_1 x} + c_2 \quad **$$

Cancelled out

$$= xe^x - e^x + c_2$$

* Note when using integration by part, we normally ignore c_1 in $g(x)$ because it will get cancelled out in the end as in **

(b) $$\int \frac{3x}{\sqrt{4-x}}\,dx = \int 3x \cdot \frac{1}{\sqrt{4-x}}\,dx$$

take this example to see how to choose which one between $3x$ & $\frac{1}{\sqrt{4-x}}$ to be $f(x)$ or $g'(x)$

- I would choose $f(x) = 3x$ & $g'(x) = \frac{1}{\sqrt{4-x}} = (4-x)^{-1/2}$

$\therefore$ from these, I'll get $f'(x) = 3$, $g(x) = \frac{(4-x)^{1/2}}{\frac{1}{2}} \times (-1)$

Note I don't add constant to $g(x)$

- $f'(x)$ looks less complicated compare to $f(x)$:) so $\int f'(x)g(x)dx$ will not be difficult.
- $g(x)$ looks equally complicated as $g'(x)$:)

$\therefore$ using integration by part

$$\int \frac{3x}{\sqrt{4-x}}\,dx = \left[3x \cdot (-2)(4-x)^{1/2}\right] - \underbrace{\int 3(-2)(4-x)^{1/2}\,dx}_{\text{Can be integrated easily}}$$

$$= -6x(4-x)^{1/2} + 6\int (4-x)^{1/2}\,dx$$

$$= -6x(4-x)^{1/2} + 6\,\frac{(4-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \times (-1) + C$$

$$= -6x(4-x)^{1/2} - 6 \times \frac{2}{3}(4-x)^{3/2} + C$$

$$= -6x(4-x)^{1/2} - 4(4-x)^{3/2} + C$$

Note if choose $f(x) = (4-x)^{-1/2}$ & $g'(x) = 3x$ will get

$\left[(4-x)^{-1/2}\left(\frac{3x^2}{2}\right)\right] - \int \frac{1}{2}(4-x)^{-3/2}\left(\frac{3x^2}{2}\right)dx \longrightarrow$ This is difficult to integrate

Chapter 7, Ex. 1

(c) $\int \frac{x}{(5x+2)^3}\,dx = \int \underbrace{x}_{f(x)} \cdot \underbrace{\frac{1}{(5x+2)^3}}_{g'(x)}$

$f'(x) = 1$ $\quad\quad g(x) = \int \frac{1}{(5x+2)^3}\,dx = \int (5x+2)^{-3}\,dx$

$g(x) = \frac{(5x+2)^{-2}}{5(-2)}$ → again I don't add a constant to $g(x)$

} try to check if you integrate correctly by diff your answer to avoid making a mistake

$$\int \frac{x}{(5x+2)^3}\,dx = \left[f(x)g(x)\right] - \int f'(x)g(x)\,dx$$

$$= x\left[\frac{(5x+2)^{-2}}{-10}\right] - \int \frac{(1)}{-10}(5x+2)^{-2}\,dx$$

$$= -\frac{x}{10(5x+2)^2} + \frac{1}{10}\int (5x+2)^{-2}\,dx$$

$$= \frac{-x}{10(5x+2)^2} + \frac{1}{10}\left[\frac{(5x+2)^{-1}}{-1(5)}\right] + c$$

$$= \frac{-x}{10(5x+2)^2} - \left(\frac{1}{50}\right)\left(\frac{1}{5x+2}\right) + c$$

Chapter 7, Ex 2.

$$I = \int \frac{1}{x} \ln x \, dx$$

$f(x)$, $g'(x)$ how to choose? If you don't know, you can just try!

Eg. choose $f(x) = \ln x$ $\qquad g'(x) = \frac{1}{x}$

$f'(x) = \frac{1}{x}$ $\qquad g(x) = \int \frac{1}{x} \, dx = \ln x$

$$\therefore I = [\ln x \cdot \ln x] - \underbrace{\int \frac{1}{x} \ln x \, dx}$$

This is exactly the same as I originally.

Hence this choice of $f(x)$, $g'(x)$ is not useful.

$\therefore f(x) = \frac{1}{x}$ $\qquad g'(x) = \ln x$ $\qquad y = \ln x$

$x = e^y$

Chapter 7, Ex. 2

(a) $I_1 = \int \underbrace{e^{2x}}_{g'(x)} \underbrace{x^2}_{f(x)} \, dx$

Note: if there is exponential function in the integrand, normally the exponential function will be chosen as $g'(x)$. This is because $g(x)$ will be in the form of exponential so it is considered not more complicated than $g'(x)$. In other words, it's not getting worse. But if you choose x^2 as $g'(x)$ in this case $\Rightarrow g(x) = \frac{x^3}{3}$ "This is considered getting worse!! more complicated!!

$g(x) = \frac{e^{2x}}{2}, \quad f'(x) = 2x$

$$\int e^{2x} x^2 \, dx = x^2 \frac{e^{2x}}{2} - \int 2x \frac{e^{2x}}{2} \, dx$$

$$= \frac{x^2 e^{2x}}{2} - \underbrace{\int x e^{2x} \, dx}_{I_2}$$

This still looks difficult to integrate but looks better than I_1

$\therefore$ Try using integration by part to I_2

$$I_2 = \int \underbrace{x}_{f(x)} \; \underbrace{e^{2x}}_{g'(x)} \, dx$$

$f'(x) = 1$:) $\quad g(x) = \frac{e^{2x}}{2}$

$$\therefore I_2 = \frac{x e^{2x}}{2} - \int (1) \frac{e^{2x}}{2} \, dx$$

$$= \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + C_1$$

$$\therefore I_1 = \frac{x^2 e^{2x}}{2} - I_2$$

$$= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4} - C_1$$

$$= \frac{1}{4}(2x^2 - 2x + 1)e^{2x} + C$$

Chapter 7, Ex. 3

$$I = \int \ln x \, dx = \int \underbrace{1}_{g'(x)} \times \underbrace{\ln x}_{f(x)} \, dx$$

Note: I choose $\ln x$ to be $f(x)$ because if choose $g'(x) = \ln x$, I will have to integrate $\ln x \Rightarrow$ difficult, no rule I can use straight away.

$\therefore \; g(x) = x \qquad f'(x) = \frac{1}{x}$

$$I = x \ln x - \int \frac{1}{x} \cdot x \, dx$$

$$= x \ln x - x + C$$

$$= x(\ln x - 1) + C$$

Ex. 4 $\quad I = \int \underbrace{x}_{f(x)} \underbrace{e^{-x}}_{g'(x)} dx \qquad g(x) = -e^{-x}$

$\because f'(x) = 1$ looks better than x

$\because g(x)$ still is in the form of exponential i.e. it doesn't look worse

$$\therefore I = -xe^{-x} - \int (1)(-e^{-x}) \, dx$$

$$= -xe^{-x} - e^{-x} + C$$

$$= -e^{-x}(1 + x) + C$$

Chapter 7, Ex. 5

$g'(y)$ $f(y)$ ∵ can diff $\ln y$ easily but if having to integrate $\ln y$ ⇒ not straight forward

$$I = \int y^3 \ln y \, dy$$

∴ $g(y) = \frac{y^4}{4}$ ⇒ get worse compare to y^3 but this is still better than having to integrate $\ln y$

$f'(y) = \frac{1}{y}$

$$I = \frac{y^4}{4} \cdot \ln y - \int \frac{1}{y} \frac{y^4}{4} \, dy$$

$$= \frac{y^4 \ln y}{4} - \int \frac{y^3}{4} \, dy$$

$$= \frac{y^4 \ln y}{4} - \frac{y^4}{4 \cdot 4} + C$$

$$= \frac{y^4 \ln y}{4} - \frac{y^4}{16} + C$$

Ex. 6 $C'(x) = 0.3x^2 + 2x$ to find $C(x)$ ∴ need to integrate $C'(x)$

$$C(x) = \int 0.3x^2 + 2x \, dx = \frac{0.3x^3}{3} + \frac{2x^2}{2} + C$$

$$= 0.1x^3 + x^2 + C \quad \text{—— (1)}$$

Given the fixed cost of 2000 pounds ⇒ this is the cost involved no matter how many items made OR in other words fixed cost = total cost when $x = 0$ or $C(0)$

∴ From (1) ⇒ $C(0) = C = 2000$

$$\therefore C(x) = 0.1x^3 + x^2 + 2000$$

$$C(20) = 0.1(20)^3 + 20^2 + 2000$$

$$= 0.1 \times 8000 + 400 + 2000$$

$$= 3200$$

Ex.7

$R(x) = \int R'(x)\,dx = \int (400 - 0.4x)\,dx$

$= 400x - \frac{0.4x^2}{2} + C$

Given that no revenue when production level is zero $\Rightarrow \therefore R(0) = 0$

$\therefore R(0) = 0 = C \Rightarrow C = 0$

Hence, $R(x) = 400x - 0.2x^2$

$R(1000) = 400 \times 1000 - 0.2 \times (1000)^2$

$= 400000 - 200000 = 200,000$

Ex.8 want to get t that results in $S(t) = 41,000$

Info given: ① $S'(t) = 60t^{1/2}$

② $S(0) = 27,000$ (because t is the time (no of weeks) after campaign began but 27,000 is no of listeners just before the campaign began)

$S(t) = \int S'(t)\,dt = \int 60t^{1/2}\,dt = 60\int t^{1/2}\,dt$

$= 60\left[\frac{t^{3/2}}{\frac{3}{2}}\right] + C = 40t^{3/2} + C$

$S(0) = 27,000 = C$

$\therefore S(t) = 40t^{3/2} + 27000 = 41,000$

$40t^{3/2} = 14000$

$t^{3/2} = 350$

$t = 350^{2/3}$

$= 49.66 = 50$ weeks

Ex. 9 Similar to Ex. 8

$B(t) = \int B'(t) = \int -6000 t^{1/3} dt$

$= -6000 \int t^{1/3} dt = -6000 \dfrac{t^{\frac{4}{3}}}{4/3} + C$

$= -4500 t^{4/3} + C$

Given the current circulation $B(0) = 640,000$ $\therefore C = 640,000$
meaning $t=0$

Find t when $B(t) = 460,000$

$\therefore 460,000 = -4500 t^{4/3} + 640,000$

$-180,000 = -4500 t^{4/3}$

$40 = t^{4/3}$

$t = 15.9 \sim 16$ weeks

Ex. 10 $f(t) = \int f'(t) dt = \int (0.004t + 0.62) dt$

$= \dfrac{0.004t^2}{2} + 0.62t + C$ million litres.

Given that at the end of 2005, that is 2 years after the beginning of 2004, $f(2) =$ 2 million litres (Note $f(1)$ = sales at the end of 2004 or beginning of 2005)

$\therefore 2 = 0.002(2)^2 + 0.62(2) + C$

$= 0.008 + 1.24 + C$

$C = 0.752$

$\therefore f(t) = 0.002t^2 + 0.62t + 0.752$

In 2020 meaning at the end of 2020 because at the beginning is the sale in 2019 $\Rightarrow t = 17$

$f(16) = 0.002(17)^2 + 0.62(17) + 0.752$

$= 0.578 + 10.54 + 0.752 = 11.87$ million litres

Ex. 11 $p(x) = \int p'(x)\,dx = \int -0.015e^{-0.01x}\,dx$

$= -0.015\int e^{u}\,dx$ where $u = -0.01x$

$du = -0.01\,dx$

$\therefore dx = \frac{1}{-0.01}\,du$

$p(x) = -0.015\int \frac{e^{u}}{-0.01}\,du$

$= 1.5\int e^{u}du$

$= 1.5\,e^{u} + c$

$= 1.5e^{-0.01x} + c$ = price - demand equation ($p - x$)

Given $x = 50$ when $p = 2.35$

$\therefore 2.35 = 1.5\,e^{-0.01(50)} + c$

$2.35 = 1.5(0.6065) + c$

$c = 1.44$

$\therefore p(x) = 1.5e^{-0.01x} + 1.44$

Next find x when $p(x) = 1.89$

$\therefore 1.89 = 1.5e^{-0.01x} + 1.44$

$\frac{1.89 - 1.44}{1.5} = e^{-0.01x}$

$\frac{0.45}{1.5} = e^{-0.01x}$

$0.3 = e^{-0.01x}$

take ln both sides

$\ln 0.3 = \ln(e^{-0.01x}) = -0.01x\ln e = -0.01x(1)$

$-1.20397 = -0.01x$

$x = 120.4$ ~ 120 packs

Ex. 12

$$P(x) = \int p'(x)\,dx = \int (165 - 0.1x)\,dx$$

$$= 165x - \frac{0.1x^2}{2} + c$$

The question asks for the change if x changes from 1500 to 1600. This is $|p(1600) - p(1500)|$. Hence, it doesn't matter if we don't know c because it will get cancelled out when finding $|p(1600) - p(1500)|$

$$p(1600) = 165(1600) - 0.1\frac{(1600)^2}{2} + c$$
$$= 136000 + c$$
$$p(1500) = 165(1500) - \frac{0.1(1500)^2}{2} + c$$
$$= 135000 + c$$

$$|p(1600) - p(1500)| = 1000$$

Ex. 13 (A) $C'(t) = R'(t)$

$$2 = 9e^{-0.5t}$$
$$\frac{2}{9} = e^{-0.5t}$$

Take ln both sides

$$\ln(2/9) = \ln(e^{-0.5t}) = -0.5t \ln e = -0.5t(1)$$
$$\ln(2/9) = -0.5t = -1.504$$
$$t = \frac{-1.504}{-0.5} = 3 \;*$$

(B) Total profit during the useful life of the game $= P(3) - P(0) = R(3) - R(0) - (C(3) - C(0))$

$$C(3) - C(0) = \int_0^3 C'(t)\,dt = \int_0^3 2\,dt = [2t]_0^3 = 6$$

$$R(3) - R(0) = \int_0^3 9e^{-0.5t}\,dt = 9\int_0^3 e^{-0.5t}\,dt = -18\left[e^{-0.5t}\right]_0^3$$
$$= -18\left[e^{-1.5} - e^0\right] = 13.984$$

$$\therefore P(3) - P(0) = 13.984 - 6 = 7.984 = 7984 \text{ Bahts}$$

Chapter 7, Ex. 17

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{r\to\infty}\left[\ln x\right]_1^r$$

$$= \lim_{r\to\infty}\left(\ln r - 0\right) = \infty$$

$\therefore$ divergent

Ex. 18 $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{r\to\infty}\left[-x^{-1}\right]_1^r$$

$$= \lim_{r\to\infty}\left[-\frac{1}{r}+1\right]$$

$$= \frac{-1}{\infty}+1$$

$$= 1$$

$\therefore$ convergent

Ex. 19 $\int x e^{-2x}\,dx$

let $f(x) = x \Rightarrow f'(x) = 1$

$g'(x) = e^{-2x} \rightarrow g(x) = \frac{e^{-2x}}{-2}$

$$\int x e^{-2x}\,dx = -\frac{x e^{-2x}}{2} - \int \frac{(1)e^{-2x}}{-2}\,dx$$

$$= \frac{-x e^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= \frac{-x e^{-2x}}{2} + \frac{1}{2}\,\frac{e^{-2x}}{-2} + C$$

$$= -\frac{x e^{-2x}}{2} - \frac{1}{4}e^{-2x} + C$$

$$\int_0^\infty x e^{-2x}\,dx = \lim_{r\to\infty}\left[-\frac{x e^{-2x}}{2} - \frac{1}{4}e^{-2x}\right]_0^\infty = -\frac{\infty e^{-2\infty}}{2} - \frac{1}{4}e^{-2\infty} + \frac{1}{4}$$

$\therefore$ divergent