

Long-run Relationship

Spurious Problem

If y_t and x_t are nonstationary series

OLS estimated result of model

$$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$$

can lead to spurious problem.

Spurious regression has significant t-statistics,
but the results are without any theoretical
meaning.

R^2 is higher than Durbin-Watson.

Long-run Relationship

Cointegrated Time-series

If y_t and x_t are nonstationary, but their linear combination $\varepsilon_t = y_t - \beta_0 - \beta_1 x_t$ is stationary.

Then, y_t and x_t are cointegrated time-series.

Cointegration Test

Statistical test that tests whether the series are cointegrated or not.

Vector Error Correction Mechanism (VECM)

From VARs unrestricted model:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix},$$

$$\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \sim IID \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right)$$

Model in matrix form:

$$X_t = A_0 + A_1 X_{t-1} + \varepsilon_t$$

Vector Error Correction Mechanism (VECM)

From: $X_t = A_0 + A_1 X_{t-1} + \epsilon_t$

Let eigenvalues and eigenvectors of A_1 be

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \vdots & \vdots \\ c_1 & c_2 \\ \vdots & \vdots \end{bmatrix}$$

Provided eigenvalues are distincts, the eigenvectors will be linear independent and C is nonsingular. Then,

$$C^{-1}A_1C = \Lambda \quad \text{and} \quad A_1 = C\Lambda C^{-1}$$

Example:

Define new vector of variables (linear combination of y_t and z_t) S_t as

$$S_t = C^{-1}X_t \quad \text{or} \quad X_t = CS_t$$

Then, $S_t = A_0^* + \Lambda S_{t-1} + \eta_t$

where: $A_0^* = C^{-1}A_0$ and $\eta_t = C^{-1}\epsilon_t$

Thus, $s_{1t} = a_{10}^* + \lambda_1 s_{1t-1} + \eta_{1t}$
 $s_{2t} = a_{20}^* + \lambda_2 s_{2t-1} + \eta_{2t}$

Example:

From

$$s_{1t} = a_{10}^* + \lambda_1 s_{1t-1} + \eta_{1t}$$
$$s_{2t} = a_{20}^* + \lambda_2 s_{2t-1} + \eta_{2t}$$

Case 1: $|\lambda_1| < 1$ and $|\lambda_2| < 1$

The two series y_t and x_t are stationary or $I(0)$.

Case 2: $\lambda_1 = 1$ and $|\lambda_2| < 1$

Series y_t is $I(0)$ and z_t is $I(1)$.

Case 3: $\lambda_1 = \lambda_2 = 1$

Series y_t and x_t are nonstationary.

Trend and Cointegration

VAR(I) with two variables

$$X_t = A_0 + A_1 X_{t-1} + \varepsilon_t$$

Assume that the two series are both $I(1)$, the model can be written in form of VECM

$$\Delta X_t = A_0 + \Pi X_{t-1} + \varepsilon_t, \quad \Pi = A_1 - I$$

where:
$$\Pi = \begin{pmatrix} \alpha_1 & -\beta\alpha_1 \\ \alpha_2 & -\beta\alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} (1 \quad -\beta)$$

Then, VECM becomes:

$$\Delta x_t = a_{10} + \alpha_1 (y_{t-1} - \beta x_{t-1}) + \varepsilon_{1t},$$

$$\Delta y_t = a_{20} + \alpha_2 (y_{t-1} - \beta x_{t-1}) + \varepsilon_{2t}$$

Trend and Cointegration

Cointegration in VAR(I) depend on the rank of Π .

Cointegration in VAR(p) models for m variables. $\Delta X_t = A_0 + \Pi X_{t-1} + \Gamma_1 \Delta X_{t-1} + \dots + \Gamma_{p-1} \Delta X_{t-p+1} + \varepsilon_t$,
 $t = p+1, \dots, n$

Cointegration depend on the rank of Π (r).

If $r=m$, all variables are stationary

If $1 \leq r \leq m-1$, there exists cointegration relations.

Trend and Cointegration

VECM can be estimated by ML.

The maximal value of the log-likelihood, for given rank r of the matrix Π , is equal to

$$\log(L_{\max}(r)) = c - \frac{n-p}{2} \sum_{j=1}^r \log(1 - \hat{\lambda}_j)$$

Eigenvalues test is to check how many of the canonical correlations differ significant from zero. $r = (\text{number of significant } \hat{\lambda}_j)$

Johansen trace test is LR-test.

$$LR(r) = 2(\log(L_{\max}(m)) - \log(L_{\max}(r))) = -(n-p) \sum_{j=r+1}^m \log(1 - \hat{\lambda}_j)$$

Trend and Cointegration

Test for number of cointegration relations

1. Test $H_0: r=0$ against $H_1: r \geq 1$
2. Test $H_0: r=1$ against $H_1: r \geq 2$
3. Iteratively test $H_0: \text{rank}(\Pi)=r$ against $H_1: \text{rank}(\Pi) \geq r+1$.

Trend and Cointegration

Trends in Johansen VECM Framework

$$\Delta X_t = \alpha(\beta X_{t-1} + \mu + \rho t) + \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + \nu + \tau t + \varepsilon_t$$

Place restrictions on the trend terms:

1. Unrestricted trend
2. Restricted trend, $\tau=0$
3. Unrestricted Constant, $\tau=0$ and $\rho=0$
4. Restricted constant, $\tau=0$, $\rho=0$, and $\nu=0$
5. No trend, $\tau=0$, $\rho=0$, $\nu=0$, and $\mu=0$