Underastanding asset demand 2: Traditional portfolio theorem

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Stock of knowledge that we knew

- VNM Expected utility
- Three types of attitude toward risk
- Certainty equivalence
- risk premium
- coefficient of risk aversion: relative v.s. absolute

Agenda

- What is traditional portfolio selection theory?
- What do we learn about investment decisions under the traditional theory?
- Demand for risky asset: determinants...

A Scenario to motivate your thoughts

- AN investor starts his day with an initial wealth of \$W0.
- The investor considers splitting/putting \$W0 into different type of assets.
- How much money should he put into risky assets / risk-free assets?
- Answering the above question is the **CORE** of the framework for optimal portfolio selection.

Terminal wealth...

Suppose with two assets; one risky and one risk-free.

rf = return on risrisk-free asset

r = return on risky asset

W0 = Initial welath

Wa = $ Invesment on risky asset

W1= (1+r) * Wa + (W0 - Wa) *(1+rf)

W1 = W0 * (1+rf)+ Wa * (r-rf)

Simplifications: Binary outcomes of return

Risky asset → "Random variable" → needs some assumptions on its statistical behavior.

GOOD ⇒ $r = rG \rightarrow$ (prob = pG)

Bad ⇒ $r = rB \rightarrow$ (prob = 1 - pG)

Assuming that:

(i) $rG > rB$ (natural assumption)
(ii) **$rG > pG*rG + (1-pG)*rB > rf > rB$ (technical assumption)**

Average return of risky-asset > return on risk-free asset

Investment outcomes

Depednings on the “realized” situation between G and B

$W1 | G = W0 * (1+rf) + Wa * (rG-rf)$

$W1 | B = W0 * (1+rf) + Wa * (rB-rf)$

Regadless to what happen, rf always stays the same.

What investor does it to choose…

Investor chooses for "Wa" that maximizes the expected utility

E[U(W1)] = pG*U(W1 | G) +(1-pG)*U(W1 | B)

EU(W1)=

pG*[U(W0*(1+rf)+Wa*(rG-rf))]+(1-pG)[U(W0*(1+rf)+Wa*(rB-rf))]

Mathemetization of Optimal risky invesment

Wa = ? Mathematically, E(U(W1)) is maximized..

Optimization

$y = f(x) \Rightarrow$ x^* is an optimizer when $f'(x^*) = 0$..

x^* is the max optimizer if $f''(x^*) < 0$ (concave)

x^* is the min optimizer if $f''(x^*) > 0$ (convex)

The first-order optimality condition

f(Wa) = EU(W1)=

pG*[U(W0*(1+rf)+Wa*(rG-rf))]+(1-pG)[U(W0*(1+rf)+Wa*(rB-rf))]

Setting the first-order condition equal to zero!

One can solve for Wa*, aka optimal investment on risky asset.

First-order condition: Interpretation

$f'(Wa) =$

$pG[U'(W0*(1+rf)+Wa*(rG-rf))]*(rG-rf) +$

$(1-pG)[U'(W0*(1+rf)+Wa*(rB-rf))]*(rB-rf) = 0$

Consider a dollar increase in risky investment:

- Foregone benefit (cost): $U'(W0*(1+rf)+Wa*(rB-rf))\ rf$
- Benefit:
  $pG[U'(W0*(1+rf)+Wa*(rG-rf))]*(rG) +$
  $(1-pG)[U'(W0*(1+rf)+Wa*(rB-rf))]*(rB)$

Analyze solution: Demand for (risky) asset

- Ultimately, one would want to study the behavior of Wa*.
- This requires an advanced tool set in math called the implicit function theory.

- To simplify things, let focus on some cases where we can solve for the solution using simple algebra.
  - Quadratic utility
  - Logarithm utility
- Then, we draw some conclusions about the general theory that

Quadratic case

$U(W) = aW - bW^2$

Recall:

$U'(*) = a - 2bW.$

$U''(*) = -2b.$

Absolute risk aversion (CARA) = $-(-2b / (a - 2bW)) = 2b/(a-2bW)$

Relative risk aversion (CRRA) = $2bW/(a-2bW)$

Note first: CARA increases as W rises; and so as CRRA.

Quadratic case: a simple case

Suppose that $U(W) = 100*W - 0.5*W^2$

$rG = 10\%$ ($pG = ½$), $rB = 0\%$

$rf = 2\%$

Solve for "Wa*" when $W0 = 10$ and $W0 = 50$.

Quadratic case: W0 = 10.

W0 =10.

W1 | G = W0 * (1+rf)+ Wa * (rG-rf) = 10*(1.02) + Wa(0.08)
W1 | B = W0 * (1+rf)+ Wa * (rB-rf) = 10*(1.02) - Wa(0.02)

U'(*) = 100 - W1.

0.5[100-(W1 | G)]*(rG-rf)+ 0.5[100-(W1 | B)]*(rB-rf) =0

Quadratic case: W0 = 10.

Rearrange terms

$[0.5*(100-10.2)*0.08 \; -0.5*(100-10.2)*(0.02)]=$

$[0.5*(0.08)^2+0.5*(0.02)^2] \; Wa$

Wa = 792.35

Risk-free allocation = 10 - 792.35 = - 782.35…

Wait!!!!!   Why do we get a NEGATIVE number?

Quadratic case: W0 = 10.

Why do we get negative invesment?

- Back to Consumption-saving problem in EE311

- Income 100 → consume 90 → Save 10
- Income 100 → consume 110 → Save -10 ➔ Borrow = 10

With the negative investment, investor is not actually investing. They are borrowing (short-sell), and use the proceeds to reinvest in the risky-asset.

Investors taking such a risky position is called a leverage-based investor.

Quadratic case: W0 = 10.

Why heavily invest in Risky-asset?

Attitude toward risk = 1/(100-10) = 1/90 so small!!!

Relatively care more on return than risk

Take larger position on upward gain. Levergae = 79 times.

Quadratic case: W0 = 50.

Oringal case when W0= 10

$[0.5*(100-10.2)*0.08 - 0.5*(100-10.2)*(0.02)]=$

$[0.5*(0.08)^2+0.5*(0.02)^2]\ Wa$

Now W0 = 50 → W0(1+rf) will get changed.*

$[0.5*(100-51)*0.08 - 0.5*(100-51)*(0.02)]=$

$[0.5*(0.08)^2+0.5*(0.02)^2]\ Wa$

Quadratic case: W0 = 50

Given our new assumption; W0 = 50 we find that

Wa = 432.35

Leverage = 8.67

Taking less on risky-position. Leverage drops.

Quadratic case: W0 = 50.

Note that:

- Wa drops from 792 to 432.
- Fraction on risky-investment drops from 79.2 to 8.67

Why? → Degree of risk tolerance varies over the level of wealth.

CA**RA** and CRR**a** rises as the investor has intial wealthy condition.

Logarithmic utility function

$U = \ln(W)$

CARA = 1/W (W rises.. CARA drops. Less risk aversion when you become wealthy.)

CRRA = 1 (constant)

Logarithmic utility function

$$\begin{aligned}&pG[1 / (W0*(1+rf)+Wa*(rG-rf))]*(rG-rf) + \\ &(1-pG)[1 / (W0*(1+rf)+Wa*(rB-rf))]*(rB-rf) =0\end{aligned}$$

$$\begin{aligned}&pG[W0*(1+rf)+Wa*(rB-rf)]*(rG-rf) = \\ &-(1-pG)[W0*(1+rf)+Wa*(rG-rf)]*(rB-rf)\end{aligned}$$

$$Wa = [(1+rf)[pG(rG-rf)+(1-pG)(rB-rf)] / (rG-rf)(rf-rB)]*W0$$

Logarithmic utility function

$$Wa = \left[(1+rf)[pG(rG-rf)+(1-pG)(rB-rf)] \,/\, (rG-rf)(rf-rB) \right] * W0$$

- [*] > 0; Wa increases as W0 inreases; decreasing CARA.
- Less aversion on risk; plenty of welath remained even if I get hurted from the investment lost.
- However, Wa/W0 is fixed.
  - B/C constant relative risk aversion
- ER =pG*rG +(1-pG)rb -rf = Excess return
  - Higher excess return attracts more investment on risky asset.
- (rG-rf)(rf-rB): return dispersions (volatility)

  \- Lowere investment on risky asset if more dispersion on return.

Conclusion

- Provide a (simple) analytically tractable framework used to under the porfolio slection.
- Built upon the language discussed in intermediate microeconomics, optimal asset investment results in demand for assets.

Conclusion

- We showed that one makes a positive investment on risky asset if
    - Excess return is negative; the more excess return, the more investment on risky-asset.
    - Leverage-based investment can be the case; VERY low CARA.
    - Higher risk reduces the demand for risky asset.
    - Risky asset can be either normal/inferior good; What types of CARA attitude it is. (+ DAra, - IAra, 0 CAra)
    - Weight onf risky investment depends on CRRA attitude; (+ DRra, - IRra, 0 CRra).