SUBJECT: EE320 – HW 5 Solutions NO: 1. DATE: 14 / Nov / 2013

① a) $\underset{x_1, x_2}{\text{Max}}\ u(x_1, x_2) = 26x_1 + 55x_2 - x_1^2 - x_1x_2 - 2x_2^2$

FONC: $\frac{\partial u}{\partial x_1} = 26 - 2x_1^* - x_2^* = 0 \quad -①$

$\frac{\partial u}{\partial x_2} = 55 - x_1^* - 4x_2^* = 0. \quad -②$

① & ② $\Rightarrow$ $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 26 \\ 55 \end{bmatrix}.$

$\Rightarrow x_1^* = \dfrac{\begin{vmatrix} 26 & 1 \\ 55 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{104 - 55}{8 - 1} = \frac{49}{7} = 7.$

$\Rightarrow x_2^* = \frac{1}{7}\begin{vmatrix} 2 & 26 \\ 1 & 55 \end{vmatrix} = \frac{110 - 26}{7} = 12.$

$\therefore (x_1^*, x_2^*) = (7, 12)$ and $u(x_1^*, x_2^*) = 421$

b) SOSC: $\frac{\partial^2 u}{\partial x_1^2} = -2 < 0$; $\frac{\partial^2 u}{\partial x_2^2} = -4 < 0$; $\frac{\partial^2 u}{\partial x_1 \partial x_2} = -1$

$|H_1| = -2 < 0$ and $|H| = \begin{vmatrix} -2 & -1 \\ -1 & -4 \end{vmatrix} = 7 > 0$

$\therefore$ SOSC for a maximum are satisfied.

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(2) Demand: $P_i = a_i - b_i Q_i$, $i = 1, 2, 3$ $(b_i > 0)$.

Cost : $TC = C_0 + CQ = C_0 + C(Q_1 + Q_2 + Q_3)$.

Firm's objective :

$$\underset{Q_1,Q_2,Q_3}{\text{Max}}\ \pi = TR_1 + TR_2 + TR_3 - TC$$

$$\Rightarrow \underset{Q_1,Q_2,Q_3}{\text{Max}}\ \pi = (a_1 - b_1Q_1)Q_1 + (a_2 - b_2Q_2)Q_2 + (a_3 - b_3Q_3)Q_3 - C_0 - C(Q_1 + Q_2 + Q_3)$$

F.O.N.C. $\pi_1 = (a_1 - C) - 2b_1 Q_1^* = 0$ — ①

$\pi_2 = (a_2 - C) - 2b_2 Q_2^* = 0$ — ②

$\pi_3 = (a_3 - C) - 2b_3 Q_3^* = 0$ — ③.

From ①, ②, & ③, we have : $Q_1^* = \dfrac{a_1 - C}{2b_1}$; $Q_2^* = \dfrac{a_2 - C}{2b_2}$; $Q_3^* = \dfrac{a_3 - C}{2b_3}$

Check sosc : $\pi_{11} = -2b_1$

$\pi_{22} = -2b_2$

$\pi_{33} = -2b_3$

$\pi_{12} = \pi_{21} = \pi_{13} = \pi_{31} = \pi_{23} = \pi_{32} = 0$.

$$\therefore H = \begin{bmatrix} -2b_1 & 0 & 0 \\ 0 & -2b_2 & 0 \\ 0 & 0 & -2b_3 \end{bmatrix}$$

$\Rightarrow |H_1| = -2b_1 < 0$

$|H_2| = \begin{vmatrix} -2b_1 & 0 \\ 0 & -2b_2 \end{vmatrix} = 4b_1b_2 > 0$

$|H_3| = |H| = -8b_1b_2b_3 < 0$.

$\therefore$ The SOSC is for a maximum profit are satisfied.

$$\pi(Q_1^*, Q_2^*, Q_3^*) = (a_1 - C)\left(\frac{a_1 - C}{2b_1}\right) - b_1\left(\frac{a_1 - C}{2b_1}\right)^2 + (a_2 - C)\left(\frac{a_2 - C}{2b_2}\right) - b_2\left(\frac{a_2 - C}{2b_2}\right)^2$$

$$+ (a_3 - C)\left(\frac{a_3 - C}{2b_3}\right) - b_3\left(\frac{a_3 - C}{2b}\right)^2 - C_0$$

$$\therefore \pi(Q_1^*, Q_2^*, Q_3^*) = \frac{(a_1 - C)^2}{4b_1} + \frac{(a_2 - C)^2}{4b_2} + \frac{(a_3 - C)^2}{4b_3} - C_0.$$

③ Demand: $P = 50 - 3q$; $q = q_1 + q_2$

Total cost: $TC_i = 25 + 5q_i$, $i = 1, 2$.

a) Before merged:

Firm 1: $\max_{q_1} \pi^1 = [50 - 3(q_1 + q_2)]q_1 - 25 - 5q_1$

$= 45q_1 - 3q_1^2 - 3q_1 q_2 - 25$

FONC: $\frac{\partial \pi^1}{\partial q_1} = 45 - 6q_1^* - 3q_2^* = 0 \Rightarrow 15 = 2q_1^* + q_2^*$ —①

Firm 2: $\max_{q_2} \pi^2 = [50 - 3(q_1 + q_2)]q_2 - 25 - 5q_2$

FONC: $\frac{\partial \pi^2}{\partial q_2} = 45 - 3q_1^* - 6q_2^* = 0 \Rightarrow 15 = q_1^* + 2q_2^*$ —②

① & ② $\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$.

$\Rightarrow q_1^* = \frac{\begin{vmatrix} 15 & 1 \\ 15 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{30 - 15}{4 - 1} = \boxed{5 = q_1^*}$; $\boxed{q_2^* = 5}$. $\Rightarrow q^* = 10$

$\Rightarrow \pi_1^* = [50 - 3(10)](5) - (25 + 25) = \underline{50}$.

Similarly, $\pi_2^* = 50$.

∴ If the firms don't merge, each will have profit of \$50.

b) After merged: q^m is the only output level.

Merged firm: $\max_{q_m} \pi^m = (50 - 3q^m)q^m - (25 + 5q^m)$

FONC: $45 - 6q^{m*} = 0 \Rightarrow \boxed{q^{m*} = 7.5}$.

Total profit of the merged firm $= \tilde{\pi}^m = [50 - 3(7.5)](7.5) - [25 + 5(7.5)]$.

$\tilde{\pi}^m = 143.75$

If the two firms split the new profit equally, each firm should get $\tilde{\pi}_i = \underline{71.875}$

∴ Since $\tilde{\pi}_i = 71.875 > \pi_i^* = 50$, the firms have incentives to merge.

⑤ $Q = f(K,L) = 8K^{1/2}L^{1/4}$

Firm: $\max_{K,L} \pi = P\cdot(8K^{1/2}L^{1/4}) - rK - wL$

F.O.N.C: $\pi_K = 4PK^{-1/2}L^{1/4} - r = 0$ — ①

$\pi_L = 2PK^{1/2}L^{-3/4} - w = 0$ — ②

① ÷ ②: $\dfrac{4PK^{-1/2}L^{1/4}}{2PK^{1/2}L^{-3/4}} = \dfrac{r}{w}$

$\dfrac{L^*}{K^*} = \dfrac{r}{2w} \Rightarrow L^* = \dfrac{r}{2w}\cdot K^*$ — ③

Sub L^* in ①:

$4PK^{*-1/2}\left(\dfrac{r}{2w}K^*\right)^{1/4} = r$

$\left(\dfrac{r}{2w}\right)^{1/4} K^{*-1/2}\times K^{*1/4} = \dfrac{r}{2P}$

$(K^*)^{-1/4\times(-4)} = \left[\dfrac{r}{2P}\times\left(\dfrac{r}{2w}\right)^{-1/4}\right]^{-4} = \dfrac{16P^4}{r^4}\cdot\dfrac{r}{2w}$

$$\boxed{K^* = \frac{8P^4}{r^3 w}}$$

Sub K^* in ③ $\Rightarrow L^* = \dfrac{r}{2w}\cdot\left(\dfrac{8P^4}{r^3w}\right) = \dfrac{4P^4}{r^2w^2} \Rightarrow$ $\boxed{r^* = \dfrac{4P^4}{r^2w^2}}$

Comparative Statics:

$\dfrac{\partial K^*}{\partial r} = \dfrac{\partial}{\partial r}\left(\dfrac{8P^4}{r^3w}\right) = -\dfrac{24P^4}{r^4w} < 0.$

$\dfrac{\partial L^*}{\partial P} = \dfrac{\partial}{\partial P}\left(\dfrac{4P^4}{r^2w^2}\right) = \dfrac{16P^3}{r^2w^2} > 0.$