Solution: Exercise 5

Extrema of functions, Mean Value Theorem, Graphing:first derivative & second derivative tests

1. Find the critical numbers of the given functions on $(-\infty, \infty)$.

   (a) $f(x) = x^3 - 3x^2 + 3x - 1$
   Solution: $f'(x) = 3x^2 - 6x + 3$

   $$f'(x) = 0 \quad \Leftrightarrow \quad 3(x^2 - 2x + 1) = 3(x-1)^2 = 0 \quad \Leftrightarrow \quad (x-1) = 0 \quad \Leftrightarrow \quad x = 1$$

   Therefore, the only critical number of $f(x)$ is $x = 1$.

   (b) $f(x) = \frac{x^2}{x^2+2}$
   Solution:

   $$f'(x) = \frac{(x^2+2)(2x) - x^2(2x)}{(x^2+2)^2} = \frac{2x^3 + 4x - x^3}{(x^2+2)^2} = \frac{x^3 + 4x}{(x^2+2)^2}$$

   $f'(x) = 0$ when $x^3 + 4x = 0 \Longrightarrow x(x^2+4) \Longrightarrow x = 0$.
   Therefore, the only critical number of $f(x)$ is $x = 0$.

   (c) $f(x) = e^{-x} + 2x$
   Solution:

   $$f'(x) = e^{-x} + 2x = -e^{-x} + 2$$

   $f'(x) = 0$ when $-e^{-x} + 2 = 0 \Longrightarrow e^{-x} = 2 \Longrightarrow e^{x} = 1/2 \Longrightarrow x = \ln(1/2)$ or $x = -\ln(2)$.
   Therefore, the only critical number of $f(x)$ is $x = -\ln(2)$.

   (d) $f(x) = -x + \sin(x)$
   Solution:

   $$f'(x) = -1 + \cos(x)$$

   $f'(x) = 0$ when $-1 + \cos(x) = 0 \Longrightarrow \cos(x) = 1 \Longrightarrow x = 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$.
   Therefore, the critical numbers of $f(x)$ are $x = 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$.

   (e) $f(x) = x^2 - 8\ln(x)$
   Solution:

   $$f'(x) = 2x - \frac{8}{x}$$

   $f'(x) = 0$ when $2x - \frac{8}{x} = 0 \Longrightarrow x = \frac{4}{x} \Longrightarrow x^2 = 4 \Longrightarrow x = \pm 2$.
   Therefore, the critical numbers of $f(x)$ are $x = -2$ and $x = 2$.

2. *Find the *absolute extrema* of each given function on the indicated interval.

   Note: The absolute extrema occur either at **critical numbers** or at **the end points** of the interval. Hence, the steps for finding the absolute extrema (maximum and minimum) of $f(x)$ on $[a, b]$ are: (i) Find all critical numbers: c_j, $j = 1, 2, \ldots, n$
   (ii) compare and find the maximum values and minimum values of $f(c_j)$, $f(a)$, and $f(b)$.

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(a) $f(x) = -x^2 + 6x$; $[1, 4]$
<u>Solution:</u> (i) We first find the critical numbers:

$$f'(x) = -2x + 6$$

$f'(x) = 0$ when $-2x + 6 = 0 \Longrightarrow x = 3$. Therefore, the critical number of $f(x)$ is $x = 3$.
(ii) Compare the function values at the critical number and endpoints (x=1,4):

$$f(1) = -(1^2) + 6(1) = 5, \qquad f(4) = -(4^2) + 6(4) = 8 \qquad f(3) = -(3^2) + 6(3) = 9$$

x	1	3	4
f(x)	5	9	8

We see from the table that
-the absolute maximum is 9 ($= f(3)$) which occurs at $x = 3$ and
-the absolute minimum is 5 ($= f(1)$) which occurs at $x = 1$.

(b) $f(x) = x^{2/3}$; $[-1, 8]$
<u>Solution:</u> (i) We first find the critical numbers:

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

and note that $f'(x) \neq 0$ for all $x \in [-1, 8]$. So critical number occurs when $f'(x)$ does not exist, i.e. when $x = 0$. Therefore, the critical number of $f(x)$ is $x = 0$.
(ii) Compare the function values at the critical number and endpoints (x=-1,8):

$$f(-1) = (-1)^{2/3} = 1, \qquad f(8) = (8)^{2/3} = 4 \qquad f(0) = 0^{2/3} = 0$$

x	-1	0	8
f(x)	1	0	4

We see from the table that
-the absolute maximum is 4 ($= f(8)$) which occurs at $x = 8$ and
-the absolute minimum is 0 ($= f(0)$) which occurs at $x = 0$.

(c) $f(x) = x^3 - 3x^2 + 3x - 1$; $[-4, 3]$

<u>Solution:</u> (i) We first find the critical numbers:

$$f'(x) = 3x^2 - 6x + 3$$

$f'(x) = 0$ when $3x^2 - 6x + 3 = 0 \Longrightarrow 3(x^2 - 2x + 1) = 3(x - 1)^2 = 0$. Therefore, the critical number of $f(x)$ is $x = 1$.
(ii) Compare the function values at the critical number and endpoints (x=-4,3):
$f(-4) = (-4)^3 - 3((-4)^2) + 3(-4) - 1 = -125$;
$f(3) = 3^3 - 3(3^2) + 3(3) - 1 = 8$;
$f(1) = 1^3 - 3(1^2) + 3(1) - 1 = 0$

x	-4	1	3
f(x)	-125	0	8

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We see from the table that
-the absolute maximum is 8 $(= f(3))$ which occurs at $x = 3$ and
-the absolute minimum is -125 $(= f(-4))$ which occurs at $x = -4$.

3. *Consider $f(x) = x^2 - 2|x|$ on the interval $[-2, 3]$. Determine (i) critical numbers; (ii) relative extrema; (iii) absolute extrema; (iv) endpoint absolute extrema.

   Solution: Note that we can write $f(x)$ as:

   - For $x < 0$, $f(x) = x^2 + 2x$
   - For $x \geq 0$, $f(x) = x^2 - 2x$.

   (i) The critical numbers:

   - For $x < 0$, $f'(x) = 2x + 2$
   - For $x > 0$, $f'(x) = 2x - 2$,

   and $f'(x) = 0$ when $2x + 2 \Rightarrow x = -1$ or $2x - 2 = 0 \Rightarrow x = 1$. Note that $f'(x)$ does not exist when $x = 0$. Hence, the critical numbers are $x = -1, 1, 0$.

   (ii) Relative extrema: Relative extrema can occur only at the critical points: $x = -1, 1, 0$. So we consider the subintervals:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of $f'(x)$	$f'(x) = 2x + 2$ $-$	$f'(x) = 2x + 2$ $+$	$f'(x) = 2x - 2$ $-$	$f'(x) = 2x - 2$ $+$

   Therefore, from the *First Derivative Test*,
   - relative maximum occurs at $x = 0$ ("+" changes to "$-$" at $x = 0$)
   and $f(0) = 0$ is the relative maximum

   - relative minima occurs at $x = -1, 1$ ("$-$" changes to "+" at $x = -1,\ 1$) and
   $f(-1) = 1^2 - 2|-1| = -1$ and $f(1) = 1^2 - 2|1| = -1$ are the relative minima.
   That is, -1 is the relative minimum, which occurs at $x = \pm 1$.

4. Determine whether the function $f(x) = x^3 + x^2$ satisfy the hypothesis of Rolle's Theorem on the interval $[-1, 0]$. If so, find all values of c that satisfy the conclusion of the theorem.

   Solution:
   Recall *Rolle's Theorem stating that:*
   Suppose f is a function that satisfies all of the following (i) -(iii):
   (i) f is continuous on the closed interval $[a, b]$.
   (ii) f is differentiable on the open interval (a, b).
   (iii) $f(a) = f(b) = 0$
   Then there is a number $c \in (a, b)$ such that $f'(c) = 0$ (I.e. f has a critical point in (a, b)).

   For (i) and (ii), since $f(x) = x^3 + x^2$ is a polynomial, it is continuous on $[-1, 0]$ and differentiable on $(-1, 0)$. For (iii), $f(-1) = (-1)^3 + (-1)^2 = 0$ and $f(0) = 0^3 + 0^2 = 0$.
   $\Rightarrow$ That is, it satisfies the hypothesis(i)-(iii) of Rolle's Theorem.

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To find c that satisfies this theorem, we first differentiate $f(x)$: $f'(x) = 3x^2 + 2x$ and solve for $c \in (-1, 0)$ from

$$f'(c) = 3c^2 + 2c = 0 \quad \Longrightarrow \quad c(3c+2) = 0 \quad \Longrightarrow \quad c = -\frac{2}{3}. \qquad \text{Note}: \quad c \neq 0 \notin (-1, 0).$$

5. *Determine whether the given function satisfy the hypothesis of the Mean Value Theorem on the indicated interval. If so, find all values of c that satisfy the conclusion of the theorem. (a)$f(x) = 1 + \sqrt{x}$; $[0, 9]$ (b) $f(x) = \frac{1}{x}$; $[-10, 10]$

Solution:
Recall *Mean Value Theorem stating that:*
Suppose f is a function that satisfies all of the following (i) -(iii):
(i) f is continuous on the closed interval $[a, b]$.
(ii) f is differentiable on the open interval (a, b).
Then there is a number $c \in (a, b)$ such that $\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$.

(a)Solution: $f(x) = 1 + \sqrt{x}$; $[0, 9]$
For (i) and (ii), $f(x) = 1 + \sqrt{x}$ is continuous on $[0, 9]$ and $f'(x) = \frac{1}{2\sqrt{x}}$ so it is differentiable on $(0, 9)$. $\Rightarrow$ That is, it satisfies the hypothesis(i)-(ii) of the Mean Value Theorem.

To find c that satisfies this theorem, solve for $c \in (0, 9)$ from

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(9) - f(0)}{9 - 0} \quad \Longrightarrow \quad \frac{1}{2\sqrt{c}} = \frac{(1 + \sqrt{9}) - (1 + \sqrt{0})}{9 - 0} = \frac{3}{9} = \frac{1}{3} \Rightarrow 2\sqrt{c} = 3 \Rightarrow c = \frac{9}{4}.$$

(b)Solution: $f(x) = \frac{1}{x}$; $[-10, 10]$
$f(x) = \frac{1}{x}$ is not defined at $0 \in [-10, 10]$, so it is not continuous on $[-10, 10]$.
$\Rightarrow$ That is, it does not satisfies the hypothesis of the Mean Value Theorem.

6. Determine the intervals on which the given function f is increasing and the interval on which f is decreasing.

(a) $f(x) = x^2 + 6x - 1$ (b) $f(x) = x^4 - 4x^3 + 9$ (c) $f(x) = x^2 e^{-x}$

Answer: The fuction $f(x)$ is increasing when $f'(x) > 0$ and $f(x)$ is decreasing when $f'(x) < 0$.

(a) $f(x) = x^2 + 6x - 1 \Longrightarrow f'(x) = 2x + 6$

$$f'(x) = 2x + 6 > 0 \quad \Longrightarrow \quad x > -3$$

so $f(x)$ is increasing on the interval $[-3, \infty)$, and

$$f'(x) = 2x + 6 < 0 \quad \Longrightarrow \quad x < -3$$

so $f(x)$ is decreasing on the interval $(\infty, -3]$.

(b) $f(x) = x^4 - 4x^3 + 9 \Longrightarrow f'(x) = 4x^3 - 12x^2$

$$f'(x) = 4x^3 - 12x^2 > 0 \quad \Longrightarrow 4x^2(x-3) > 0 \quad \Longrightarrow x > 3$$

so $f(x)$ is increasing on the interval $[3, \infty)$, and

$$f'(x) = 4x^3 - 12x^2 < 0 \quad \Longrightarrow 4x^2(x-3) < 0 \quad \Longrightarrow x < 3$$

so $f(x)$ is decreasing on the interval $(\infty, 3]$.

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(c) $f(x) = x^2e^{-x} \Longrightarrow f'(x) = -x^2e^{-x} + 2xe^{-x} \Longrightarrow f'(x) = xe^{-x}(2-x)$,
since $e^{-x} > 0$, $f'(x) = 0 \Longrightarrow x = 0$ or $x = 2$:

	$x < 0$	$0 < x < 2$	$2 < x$
Sign of $f'(x) = xe^{-x}(2-x)$	$-$	$+$	$-$
$(x)(e^{-x})(2-x)$	$(-)(+)(+)$	$(+)(+)(+)$	$(+)(+)(-)$

so $f(x)$ is increasing ($f'(x) = +$) on the interval $[0, 2]$,
and $f(x)$ is decreasing ($f'(x) = -$) on the interval $(\infty, 0] \cup [2, \infty)$.

7. *Use the *First Derivative Test* to find the relative extrema of the given function.
   (a) $f(x) = x^3 - 3x$ (b) $f(x) = x^3 + x - 3$ (c) $\frac{x^2+3}{x+1}$

Solution:

(a) $f(x) = x^3 - 3x \Longrightarrow f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$,
$f'(x) = 0 \Longrightarrow x = \pm 1$ are critical numbers:

	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of $f'(x) = 3(x-1)(x+1)$	$+$	$-$	$+$
$(x-1)(x+1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = -1$ ("+" changes to "−" at $x = -1$)
and $f(-1) = (-1)^3 - 3(-1) = 2$ is the relative maximum
- relative minimum occurs at $x = 1$ ("−" changes to "+" at $x = 1$) and
$f(1) = 1^3 - 3(1) = -2$ is the relative minimum.

(b) $f(x) = x^3 + x - 3 \Longrightarrow f'(x) = 3x^2 + 1 \neq 0, \forall x \in (-\infty, \infty)$.

$f'x)$ is well-defined for any x and $f'(x) \neq 0$ imply that there is *no critical number* for $f(x)$. Since a relative extremum can only occur at a critical number and there is no critical number, therefore, we conclude that **there is no relative extrema** for this function.

(c) $f(x) = \frac{x^2+3}{x+1}$ Note that the domain of this function is all real number except for $x = -1$: $D = \mathbb{R}/\{-1\}$
$\Longrightarrow f'(x) = \frac{(x+1)(2x)-(x^2+3)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$,
$f'(x) = 0 \Longrightarrow (x+3)(x-1) = 0 \quad \Longrightarrow x = -3, 1$ are the critical numbers.

	$x < -3$	$-3 < x < 1 (x \neq -1)$	$1 < x$
Sign of $f'(x)$	$+$	$-$	$+$
$(x+3)(x-1)$	$(-)(-)$	$(+)(-)$	$(+)(+)$

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = -3$ ("+" changes to "−" at $x = -3$)
and $f(-3) = \frac{(-3)^2+3}{-3+1} = -6$ is the relative maximum
- relative minimum occurs at $x = 1$ ("−" changes to "+" at $x = 1$) and
$f(1) = \frac{1^2+3}{1+1} = 2$ is the relative minimum.

Notice that it is possible to have a " relative" minimum *larger* than a " relative" maximum.

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8. Find values of a, b, and c such that $f(x) = ax^2 + bx + c$ has a relative maximum 6 at $x = 2$ and the graph of f has y-intercepts at $y = 4$.

   Solution:
   - For y-intercepts at $y = 4$, $x = 0 \;\Rightarrow\; 4 = y = f(0) \;\Rightarrow\; 4 = a0^2 + b(0) + c \;\Rightarrow\; c = 4$.

   $f(x) = ax^2 + bx + c \Longrightarrow f'(x) = 2ax + b$.
   Critical number: $f'(x) = 2ax + b = 0 \Rightarrow x = -\frac{b}{2a}$. Since a relative maximum has to occur a critical number:
   $$-\frac{b}{2a} = 2 \quad\Rightarrow\; b = -4a.$$
   - Using $y = 6$ when $x = 2$, and the fact from above that $c = 4$, $b = -4a$
   $$6 = f(2) = a\;2^2 + 2b + c \;\Rightarrow\; 6 = 4a + 2b + 4 \Rightarrow 1 = 2a + b \quad\Rightarrow 1 = 2a - 4a \;\Rightarrow\; a = -\frac{1}{2}.$$
   Therefore, $b = -4(-1/2) = 2$. That is, $a = -\frac{1}{2}$, $b = 2$, $c = 4$.

9. *For each given function,
   (i) use the Second Derivative Test, when applicable, to find the relative extrema;
   (ii) find intercepts and points of inflection, when possible;
   (iii) find the intervals on which it is increasing and the interval on which it is decreasing;
   (iv) find the intervals on which it is concave up and the intervals on which it is concave down; and **(v)** sketch the graph.

   (a) $f(x) = x^3 + 3x^2 + 3x + 1$
   (b) $f(x) = 6x^5 - 10x^3$
   (c) $f(x) = \frac{x}{x^2+2}$
   (d) $f(x) = 2x - x\ln(x)$

   Solution:

   Note that there are 2 steps
   (1) Find critical numbers c_j
   (2) Check sign of $f''(c_j)$:
   - If $f''(c_j) < 0$, $f(c_j)$ is a relative maximum.
   - If $f''(c_j) > 0$, $f(c_j)$ is a relative minimum.
   (If $f''(c_j) = 0$, no conclusion – use the First Derivative Test.)

   (a) $f(x) = x^3 + 3x^2 + 3x + 1$
   $$f'(x) = 3x^2 + 6x + 3, \qquad f''(x) = 6x + 6.$$
   (i) Critical numbers:
   $$f'(x) = 3x^2 + 6x + 3 = 0 \quad\Rightarrow\quad 3(x+1)^2 = 0 \quad\Rightarrow\quad x = -1 \text{ is the only critical number.}$$
   From the Second Derivative Test,
   $$f''(-1) = 6(-1) + 6 = 0$$
   so we cannot conclude anything. The First derivative will be used instead,

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	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f'(x) = 3(x+1)^2$	+	+

Therefore, from the table above, the *First Derivative Test* implies that there is **no relative extremum** since the sign of $f'(x)$ is always positive.

(ii) Find intercepts and points of inflection:
- The y-intercept(x=0) is at $y = f(0) = 0^3 + 30^2 + 3(0) + 1 = 1 \Rightarrow \; y = 1$.
- The x-intercept(y=0) is at $0 = f(x) = (x+1)^3 \Rightarrow \; x = -1$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 6x + 6 = 0 \qquad \Longrightarrow \qquad x = -1.$$

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f''(x) = 6(x+1)$	$-$	+

Since the sign of $f''(x)$ changes at the $x = -1$, it gives an inflection point. When $x = -1$, $y = f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0$. Hence, the inflection point is $(-1, 0)$.
(iii) Since $f'(x) = 3(x+1)^2 > 0$for all $x \neq -1$, and $f'(x) = 0$ at $x = -1$, $f(x)$ is increasing on $(-\infty, \infty)$.
(iv) Concavity:

$$f''(x) = 6x + 6 > 0 \;\Rightarrow\; x > -1 \text{ and } \quad f''(x) = 6x + 6 < 0 \;\Rightarrow\; x < -1.$$

That is, $f(x)$ is concave up on $(-1, \infty)$ and $f(x)$ is concave down on $(-\infty, -1)$.

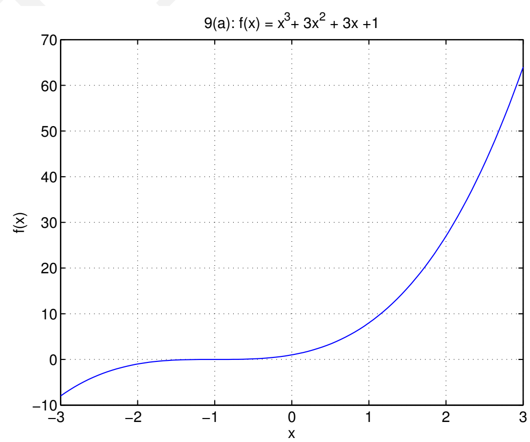

Figure 1: 9 (a): $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = 6x^5 - 10x^3$

(i) Critical numbers:

$$f'(x) = 30x^4 - 30x^2 = 30x^2(x^2-1) = 30x^2(x-1)(x+1) = 0 \;\Rightarrow$$

$x = 0, -1, 1$ are the only critical numbers. From

$$f''(x) = 120x^3 - 60x = 60x(2x^2-1)$$

and from the Second Derivative Test,
$f''(0) = 0 \Longrightarrow$, no conclusion
$f''(-1) = -60 < 0 \Longrightarrow$, $f(-1) = -6 + 10 = 4$ is a relative maximum
$f''(1) = 60 > 0 \Longrightarrow$, $f(1) = 6 - 10 = -4$ is a relative minimum.
The First Derivative Test will be used to identify the critical number $x = 0$ as shown in the table. Since the sign of $f'(x)$ does not change at $x = 0$, $x = 0$ does not give an extremum.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
Sign of $f'(x) = 30x^2(x-1)(x+1)$	$+$	$-$	$-$	$+$
$x^2(x-1)(x+1)$	(+)(−)(−)	(+)(−)(+)	(+)(−)(+)	(+)(+)(+)

(ii) Find intercepts and points of inflection:
- The y-intercept(x=0) is at $y = f(0) = 6(0)^5 - 10(0)^3 = 0 \Rightarrow\; y = 0$.
- The x-intercept(y=0) occurs when $0 = f(x) = x^3(6x^2 - 10)$
$\Rightarrow$ at $x = 0$, $x = -\sqrt{5/3}$, $x = \sqrt{5/3}$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = 60x(2x^2-1) = 0 \qquad \Longrightarrow \qquad x = 0, -\sqrt{1/2}, \sqrt{1/2}.$$

	$x \in (-\infty, -\sqrt{\frac{1}{2}})$	$x \in (-\sqrt{\frac{1}{2}}, 0)$	$x \in (0, \sqrt{\frac{1}{2}})$	$x \in (\sqrt{\frac{1}{2}}, \infty)$
Sign of $f''(x) = 60x(2x^2-1)$	$-$	$+$	$-$	$+$
$(x)(x-\sqrt{1/2})(x+\sqrt{1/2})$	(−)(−)(+)	(−)(−)(+)	(+)(−)(+)	(+)(+)(+)

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{1/2}, \sqrt{1/2}$, they give inflection points.
Since $f(0) = 0$, $f(-\sqrt{1/2}) = 7\sqrt{2}/4$, $f(\sqrt{1/2}) = -7\sqrt{2}/4$,
the inflection points are $(0, 0)$, $(-\sqrt{1/2}, 7\sqrt{2}/4)$,$(\sqrt{1/2}, -7\sqrt{2}/4)$.

(iii) From the table in (i), we see that
$f(x)$ is increasing ($f'(x) > 0$) on $(-\infty, -1) \cup (1, \infty)$ and
$f(x)$ is decreasing ($f'(x) < 0$) on $(-1, 1)$.

(iv) From the table in (ii), we see that
$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$ and
$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$.

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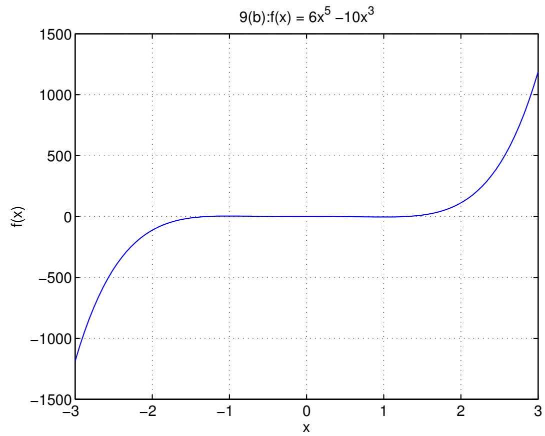

Figure 2: 9 (b): $f(x) = 6x^5 - 10x^3$

(c) $f(x) = \frac{x}{x^2+2}$

(i) Critical numbers:

$$f'(x) = \frac{(x^2+2) - 2x^2}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2} = 0 \;\Rightarrow\; \quad 2 - x^2 = 0$$

$x = -\sqrt{2}, \sqrt{2}$ are the only critical numbers. From

$$f''(x) = \frac{2x(x^2+2)(x^2-6)}{(x^2+2)^4} = \frac{2x(x^2-6)}{(x^2+2)^3}$$

and from the Second Derivative Test,

$f''(-\sqrt{2}) = \frac{\sqrt{2}}{8} > 0 \Longrightarrow, f(-\sqrt{2}) = \frac{-\sqrt{2}}{(-\sqrt{2})^2+2} = -\sqrt{2}/4$ is a relative minimum

$f''(\sqrt{2}) = -\frac{\sqrt{2}}{8} < 0 \Longrightarrow, f(\sqrt{2}) = \frac{\sqrt{2}}{(\sqrt{2})^2+2} = \sqrt{2}/4$ is a relative maximum.

(ii) Find intercepts and points of inflection:

- The y-intercept(x=0) is at $y = f(0) = 0 \Rightarrow \; y = 0$.
- The x-intercept(y=0) occurs when $0 = f(x) \Rightarrow$ at $x = 0$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$:

$$f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3} = 0 \qquad \Longrightarrow \quad x = 0, -\sqrt{6}, \sqrt{6}.$$

Since the denominator $(x^2+2)^3 > 0$, the sign of $f''(x)$ depends on the numerator $2x(x^2 - 6) = 2x(x-\sqrt{6})(x+\sqrt{6})$ as shown in the table below.

	$x \in (-\infty, -\sqrt{6})$	$x \in (-\sqrt{6}, 0)$	$x \in (0, \sqrt{6})$	$x \in (\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3}$ $(2x)(x-\sqrt{6})(x+\sqrt{6})$	$-$ $(-)(-)(+)$	$+$ $(-)(-)(+)$	$-$ $(+)(-)(+)$	$+$ $(+)(+)(+)$

Since the sign of $f''(x)$ changes at the $x = 0, -\sqrt{6}, \sqrt{6}$, they give inflection points. Since $f(0) = 0$, $f(-\sqrt{6}) = \frac{-\sqrt{6}}{(-\sqrt{6})^2+2} = -\sqrt{6}/8$, $f(\sqrt{1/2}) = \frac{\sqrt{6}}{(\sqrt{6})^2+2} = \sqrt{6}/8$,
the inflection points are $(0,0)$, $(-\sqrt{6}, -\sqrt{6}/8)$,$(\sqrt{6}, \sqrt{6}/8)$.

(iii) From

$$f'(x) = \frac{2 - x^2}{(x^2+2)^2} = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2},$$

the denominator is always positive, so the sign of $f'(x)$ depends on the numerator: $(\sqrt{2}-x)(\sqrt{2}+x)$.

	$x \in (-\infty, -\sqrt{2})$	$x \in (-\sqrt{2}, \sqrt{2})$	$x \in (\sqrt{2}, \infty)$
Sign of $f'(x) = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2}$ $(\sqrt{2}-x)(\sqrt{2}+x)$	$-$ $(+)(-)$	$+$ $(+)(+)$	$-$ $(-)(+)$

Hence, $f(x)$ is increasing ($f'(x) > 0$) on $(-\sqrt{2}, \sqrt{2})$ and
$f(x)$ is decreasing ($f'(x) < 0$) on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

(iv) From the table in (ii), we see that
$f(x)$ is concave up ($f''(x) > 0$) on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$ and
$f(x)$ is concave down ($f''(x) < 0$) on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$.

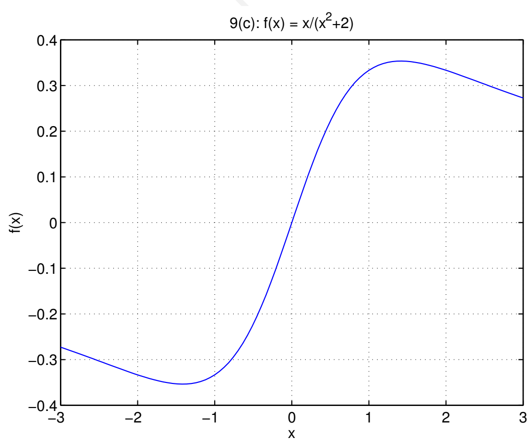

Figure 3: 9 (c): $f(x) = \frac{x}{x^2+2}$

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(d) $f(x) = 2x - x\ln(x)$
First note that the domain of f is

$$\mathcal{D} = (0, \infty),$$

since $\ln(x)$ is only well-defined for $x > 0$.
(i) Critical numbers: $f'(x) = 2 - \left[x\frac{1}{x} + \ln(x)\right] = 1 - \ln(x)$.

$$f'(x) = 1 - \ln(x) = 0 \;\Rightarrow\; \ln(x) = 1 \;\Rightarrow\; x = e^1.$$

$x = e$ is the only critical number. From

$$f''(x) = -\frac{1}{x}$$

and from the Second Derivative Test,
$f''(e) = -\frac{1}{e} < 0 \Longrightarrow$, $f(e) = 2e - e\ln(e) = 2e - e = e$ is a relative maximum.

(ii) Find intercepts and points of inflection:
- There is no y-intercept, since $x = 0$ is not in the domain of the function $f(x) = 2x - x\ln(x)$ ($\ln(x)$ is not well-defined at $x = 0$).
- The x-intercept(y=0) occurs when $0 = f(x) \Rightarrow 2x - x\ln(x) = x(2 - \ln(x)) = 0 \Rightarrow x = 0$ or $\ln(x) = 2 \Rightarrow$ at $x = 0, x = e^2$.
- To find the inflection points, we first solve $f''(x) = 0$ and see the sign change of $f''(x)$: notice that

$$f''(x) = -\frac{1}{x} \neq 0 \qquad \Longrightarrow \qquad \text{There is no inflection point.}$$

(iii) From $f'(x) = 1 - \ln(x)$,

- $f'(x) > 0$ when $1 - \ln(x) > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \Rightarrow f(x)$ is **increasing** on $(-\infty, e]$.
- $f'(x) < 0$ when $1 - \ln(x) < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \Rightarrow f(x)$ is **decreasing** on $[e, \infty)$.

(iv) From $f''(x) = -\frac{1}{x}$,
$f(x)$ is concave down ($f''(x) < 0 \Leftrightarrow x > 0$) on $(0, \infty)$. Note that $f(x)$ is not concave up, since $f''(x) > 0 \Leftrightarrow x < 0$, but $x < 0$ is not in the domain of $f(x)$: $\mathcal{D} = (0, \infty)$.

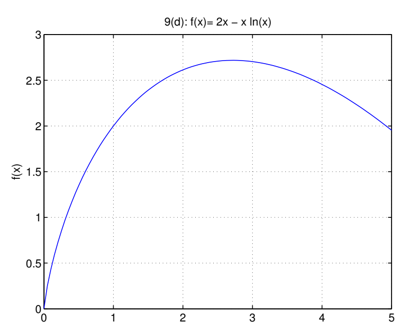

Figure 4: 9 (d): $f(x) = 2x - x\ln(x)$