9.1

(a)

$$\mathcal{L} = (x + 2)(y + 1) + \lambda(20 - ax - 5y)$$

$$\mathcal{L}_x = y + 1 - \lambda a = 0$$

$$\mathcal{L}_y = x + 2 - 5\lambda = 0$$

$$\mathcal{L}_\lambda = 20 - ax - 5y = 0$$

(b)

$$\mathcal{L} = xyz + \lambda(5 - ax - y - z)$$

$$\mathcal{L}_x = yz - \lambda a = 0$$

$$\mathcal{L}_y = xz - \lambda = 0$$

$$\mathcal{L}_z = xy - \lambda = 0$$

$$\mathcal{L}_\lambda = 5 - ax - y - z = 0$$

(c)

$$\mathcal{L} = x^{0.5}y^{0.5} + \lambda(g_0 - ax - y)$$

$$\mathcal{L}_x = 0.5x^{-0.5}y^{0.5} - \lambda a = 0$$

$$\mathcal{L}_y = 0.5x^{0.5}y^{-0.5} - \lambda = 0$$

$$\mathcal{L}_\lambda = g_0 - ax - y = 0$$

(d)

$$\mathcal{L} = A\ln(x^\alpha y^\beta z^\gamma) + \lambda(g_0 - ax^2 - y^2 - z^2)$$

$$= A(\alpha\ln x + \beta\ln y + \gamma\ln z) + \lambda(g_0 - ax^2 - y^2 - z^2)$$

$$\mathcal{L}_x = \frac{A\alpha}{x} - 2ax\lambda = 0$$

$$\mathcal{L}_y = \frac{A\beta}{y} - 2y\lambda = 0$$

$$\mathcal{L}_z = \frac{A\gamma}{z} - 2z\lambda = 0$$

$$\mathcal{L}_\lambda = g_0 - ax^2 - y^2 - z^2 = 0$$

(e) $$\mathcal{L} = x^2 + y^2 + z^2 + \lambda(g_0 - ax - y - z)$$

$$\mathcal{L}_x = 2x - \lambda a = 0$$

$$\mathcal{L}_y = 2y - \lambda = 0$$

$$\mathcal{L}_z = 2z - \lambda = 0$$

$$\mathcal{L}_\lambda = g_0 - ax - y - z = 0$$

(f) $$\mathcal{L} = x - y + \lambda\left(g_0 - ax^{0.5}y^{0.5}\right)$$

$$\mathcal{L}_x = 1 - 0.5\lambda a x^{-0.5}y^{0.5} = 0$$

$$\mathcal{L}_y = -1 - 0.5\lambda a x^{0.5}y^{-0.5} = 0$$

$$\mathcal{L}_\lambda = g_0 - ax^{0.5}y^{0.5} = 0$$

(g) $$\mathcal{L} = ax + y + z + \lambda\left(g_0 - A\ln(x^\alpha y^\beta z^\gamma)\right)$$

$$= ax + y + z + \lambda\left(g_0 - A(\alpha\ln x + \beta\ln y + \gamma\ln z)\right)$$

$$\mathcal{L}_x = a - \lambda A\alpha x^{-1} = 0$$

$$\mathcal{L}_y = 1 - \lambda A\beta y^{-1} = 0$$

$$\mathcal{L}_z = 1 - \lambda A\gamma z^{-1} = 0$$

$$\mathcal{L}_\lambda = g_0 - A(\alpha\ln x + \beta\ln y + \gamma\ln z) = 0$$

9.2

(a)

$$\bar{H} = \begin{bmatrix} 0 & 1 & -a \\ 1 & 0 & -5 \\ -a & -5 & 0 \end{bmatrix}$$

$$\left|\bar{H}\right| = (-1)\begin{vmatrix} 1 & -a \\ -5 & 0 \end{vmatrix} - a\begin{vmatrix} 1 & -a \\ 0 & -5 \end{vmatrix} = 5a + 5a = 10a > 0$$

(b)

$$\bar{H} = \begin{bmatrix} 0 & z & y & -a \\ z & 0 & x & -1 \\ y & x & 0 & -1 \\ -a & -1 & -1 & 0 \end{bmatrix}$$

$$\left|\bar{H}_2\right| = \begin{vmatrix} 0 & z & -a \\ z & 0 & -1 \\ -a & -1 & 0 \end{vmatrix} = -z(-a) - a(-z) = 2az > 0$$

$$\left|\bar{H}\right| = -z\begin{vmatrix} z & y & -a \\ x & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} + y\begin{vmatrix} z & y & -a \\ 0 & x & -1 \\ -1 & -1 & 0 \end{vmatrix} + a\begin{vmatrix} z & y & -a \\ 0 & x & -1 \\ x & 0 & -1 \end{vmatrix}$$

$$= -z(-y(-1) + (-z + ax)) + y(z(-1) - y(-1) - ax) + a(z(-x) + x(-y + ax))$$

$$= -zy + z^2 - axz - yz + y^2 - axy - axz - axy + a^2x^2$$

$$= a^2x^2 + y^2 + z^2 - 2yz - 2axz - 2axy, \quad \text{which should be} < 0 \text{ for a max}$$

The second-order conditions are not satisfied globally, but are satisfied at the values of x, y, and z that solve the first-order conditions (see answer to problem (9.1a): $x^* = y^* = z^* = 5/3$:

$$\left|\bar{H}\right| = \frac{25}{9} - \frac{50}{9} + \frac{25}{9} + \frac{25}{9} - \frac{50}{9} - \frac{50}{9} = \frac{-75}{9} < 0.$$

(c)

$$\bar{H} = \begin{bmatrix} -0.25x^{-\frac{3}{2}}y^{\frac{1}{2}} & 0.25x^{-\frac{1}{2}}y^{-\frac{1}{2}} & -a \\ 0.25x^{-\frac{1}{2}}y^{-\frac{1}{2}} & -0.25x^{\frac{1}{2}}y^{-\frac{3}{2}} & -1 \\ -a & -1 & 0 \end{bmatrix}$$

$$\left|\bar{H}\right| = (-a)\begin{vmatrix} 0.25x^{-\frac{1}{2}}y^{-\frac{1}{2}} & -a \\ -0.25x^{\frac{1}{2}}y^{-\frac{3}{2}} & -1 \end{vmatrix} + \begin{vmatrix} -0.25x^{-\frac{3}{2}}y^{\frac{1}{2}} & -a \\ 0.25x^{-\frac{1}{2}}y^{-\frac{1}{2}} & -1 \end{vmatrix}$$

$$= -a\left(-0.25x^{-\frac{1}{2}}y^{-\frac{1}{2}} - 0.25ax^{\frac{1}{2}}y^{-\frac{3}{2}}\right) + \left(0.25x^{-\frac{3}{2}}y^{\frac{1}{2}} + 0.25x^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)$$

$$= (-)((-) - (+)) + ((+) + (+)) > 0$$

(d)

$$\bar{H} = \begin{bmatrix} -\frac{A\alpha}{x^2} - 2a\lambda & 0 & 0 & -2ax \\ 0 & -\frac{A\beta}{y^2} - 2\lambda & 0 & -2y \\ 0 & 0 & -\frac{A\gamma}{z^2} - 2\lambda & -2z \\ -2ax & -2y & -2z & 0 \end{bmatrix}$$

$$\left|\bar{H}_2\right| = \begin{vmatrix} -\frac{A\alpha}{x^2} - 2a\lambda & 0 & -2ax \\ 0 & -\frac{A\beta}{y^2} - 2\lambda & -2y \\ -2ax & -2y & 0 \end{vmatrix}$$

$$= \left(-\frac{A\alpha}{x^2} - 2a\lambda\right)\left(-4y^2\right) - 2ax\left(-(-2ax)\left(-\frac{A\beta}{y^2} - 2\lambda\right)\right)$$

$$= (-)(-) - (+)(-(-)(-)) > 0$$

$$\left|\bar{H}\right| = \left(-\frac{A\alpha}{x^2} - 2a\lambda \right) \begin{vmatrix} -\frac{A\beta}{y^2} - 2\lambda & 0 & -2y \\ 0 & -\frac{A\gamma}{z^2} - 2\lambda & -2z \\ -2y & -2z & 0 \end{vmatrix} + 2ax \begin{vmatrix} 0 & 0 & -2ax \\ -\frac{A\beta}{y^2} - 2\lambda & 0 & -2y \\ 0 & -\frac{A\gamma}{z^2} - 2\lambda & -2z \end{vmatrix}$$

$$= \left(-\frac{A\alpha}{x^2} - 2a\lambda \right) \left(\left(-\frac{A\beta}{y^2} - 2\lambda \right) \left(-4z^2 \right) - 2y \left(0 - (-2y) \left(-\frac{A\gamma}{z^2} - 2\lambda \right) \right) \right)$$

$$+ 2ax \left((-2ax) \left(-\frac{A\beta}{y^2} - 2\lambda \right) \left(-\frac{A\gamma}{z^2} - 2\lambda \right) \right)$$

$$= (-)((+) - (-)) + (+)((-)(-)(-)) < 0$$

(e)

$$\bar{H} = \begin{bmatrix} 2 & 0 & 0 & -a \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -a & -1 & -1 & 0 \end{bmatrix}$$

$$\left|\bar{H}_2\right| = \begin{vmatrix} 2 & 0 & -a \\ 0 & 2 & -1 \\ -a & -1 & 0 \end{vmatrix} = 2(-1) - a(2a) < 0$$

$$\left|\bar{H}\right| = 2 \begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{vmatrix} + a \begin{vmatrix} 0 & 0 & -a \\ 2 & 0 & -1 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= 2(2(-1) - (2)) + a(-2(2a)) = -8 - 4a^2 < 0$$

(f)

$$\bar{H} = \begin{bmatrix} 0.25a\lambda x^{-1.5}y^{0.5} & -0.25a\lambda x^{-0.5}y^{-0.5} & -0.5ax^{-0.5}y^{0.5} \\ -0.25a\lambda x^{-0.5}y^{-0.5} & 0.25a\lambda x^{0.5}y^{-1.5} & -0.5ax^{0.5}y^{-0.5} \\ -0.5ax^{-0.5}y^{0.5} & -0.5ax^{0.5}y^{-0.5} & 0 \end{bmatrix}$$

$$|\bar{H}| = -0.5ax^{-0.5}y^{0.5}\left(0.125a^2\lambda y^{-1} + 0.125a^2\lambda y^{-1}\right)$$

$$+ \ 0.5ax^{0.5}y^{-0.5}\left(-0.125a^2\lambda x^{-1} - 0.125a^2\lambda x^{-1}\right) < 0$$

$$\bar{H} = \begin{bmatrix} \lambda\frac{A\alpha}{x^2} & 0 & 0 & -\frac{A\alpha}{x} \\ 0 & \lambda\frac{A\beta}{y^2} & 0 & -\frac{A\beta}{y} \\ 0 & 0 & \lambda\frac{A\gamma}{z^2} & -\frac{A\gamma}{z} \\ -\frac{A\alpha}{x} & -\frac{A\beta}{y} & -\frac{A\gamma}{z} & 0 \end{bmatrix}$$

$$|\bar{H}_2| = \begin{vmatrix} \lambda\frac{A\alpha}{x^2} & 0 & -\frac{A\alpha}{x} \\ 0 & \lambda\frac{A\beta}{y^2} & -\frac{A\beta}{y} \\ -\frac{A\alpha}{x} & -\frac{A\beta}{y} & 0 \end{vmatrix}$$

$$= \left(\lambda\frac{A\alpha}{x^2}\right)\left(-\frac{A^2\beta^2}{y^2}\right) - \frac{A\alpha}{x}\left(\lambda\frac{A^2\alpha\beta}{xy^2}\right) < 0$$

$$\left|\bar{H}\right| = \lambda\frac{A\alpha}{x^2}\begin{vmatrix} \lambda\frac{A\beta}{y^2} & 0 & -\frac{A\beta}{y} \\ 0 & \lambda\frac{A\gamma}{z^2} & -\frac{A\gamma}{z} \\ -\frac{A\beta}{y} & -\frac{A\gamma}{z} & 0 \end{vmatrix} + \frac{A\alpha}{x}\begin{vmatrix} 0 & 0 & -\frac{A\alpha}{x} \\ \lambda\frac{A\beta}{y^2} & 0 & -\frac{A\beta}{y} \\ 0 & \lambda\frac{A\gamma}{z^2} & -\frac{A\gamma}{z} \end{vmatrix}$$

$$= \lambda\frac{A\alpha}{x^2}\left(\left(\lambda\frac{A\beta}{y^2}\right)\left(-\frac{A^2\gamma^2}{z^2}\right) - \frac{A\beta}{y}\left(\lambda\frac{A^2\beta\gamma}{yz^2}\right)\right) + \frac{A\alpha}{x}\left(\left(-\lambda\frac{A\beta}{y^2}\right)\left(\lambda\frac{A^2\alpha\gamma}{xz^2}\right)\right) < 0$$

9.3

(a) $$\bar{H}\begin{bmatrix} \partial x^*/\partial a \\ \partial y^*/\partial a \\ \partial \lambda^*/\partial a \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ x \end{bmatrix}$$

$$\frac{\partial x^*}{\partial a} = \frac{\begin{vmatrix} \lambda & 1 & -a \\ 0 & 0 & -5 \\ x & -5 & 0 \end{vmatrix}}{|\bar{H}|} = \frac{\lambda(-25) + x(-5)}{|\bar{H}|} = \frac{(-)}{(+)} < 0 \quad \text{since } |\bar{H}| > 0 \text{ by SOC.}$$

$$\frac{\partial y^*}{\partial a} = \frac{\begin{vmatrix} 0 & \lambda & -a \\ 1 & 0 & -5 \\ -a & x & 0 \end{vmatrix}}{|\bar{H}|} = \frac{(-1)(ax) - a(-5\lambda)}{|\bar{H}|}$$

$$= \frac{-ax + 5a\lambda}{|\bar{H}|} = \frac{-5a\lambda + 2a + 5a\lambda}{|\bar{H}|} \quad \text{since } x = 5\lambda - 2 \text{ by FOC}$$

$$= \frac{2a}{|\bar{H}|} = \frac{(+)}{(+)} > 0$$

(b) $$\bar{H}\begin{bmatrix} \partial x^*/\partial a \\ \partial y^*/\partial a \\ \partial z^*/\partial a \\ \partial \lambda^*/\partial a \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ 0 \\ x \end{bmatrix}$$

$$\frac{\partial x^*}{\partial a} = \frac{\begin{vmatrix} \lambda & z & y & -a \\ 0 & 0 & x & -1 \\ 0 & x & 0 & -1 \\ x & -1 & -1 & 0 \end{vmatrix}}{|\bar{H}|} = \frac{\lambda\begin{vmatrix} 0 & x & -1 \\ x & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} - x\begin{vmatrix} z & y & -a \\ 0 & x & -1 \\ x & 0 & -1 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{\lambda(-x(-1) - (-x)) - x(z(-x) + x(-y+ax))}{|\bar{H}|} = \frac{\lambda x + \lambda x + zx^2 + x^2y - x^3a}{|\bar{H}|}$$

$$= \frac{2\lambda x + zx^2}{|\bar{H}|} \quad \text{since } ax = y \text{ by FOC} \Rightarrow x^2y = ax^3$$

$$= \frac{(+)}{(-)} < 0 \quad \text{since } |\bar{H}| < 0 \text{ by SOC.}$$

$$\frac{\partial y^*}{\partial a} = \frac{\begin{vmatrix} 0 & \lambda & y & -a \\ z & 0 & x & -1 \\ y & 0 & 0 & -1 \\ -a & x & -1 & 0 \end{vmatrix}}{|\bar{H}|} = \frac{y\begin{vmatrix} \lambda & y & -a \\ 0 & x & -1 \\ x & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & \lambda & y \\ z & 0 & x \\ -a & x & -1 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{y(-\lambda) + x(-y+ax)) + (-z(-\lambda - xy) - a(\lambda x))}{|\bar{H}|}$$

$$= \frac{-\lambda y - xy^2 + ax^2y + z\lambda + xyz - \lambda ax}{|\bar{H}|}$$

$$= \frac{xyz - \lambda ax}{|\bar{H}|} \quad \text{since } -xy^2 = ax^2y \text{ and } \lambda z = \lambda y \text{ by FOC}$$

$$= \frac{xyz - \lambda ax}{|\bar{H}|} = 0 \text{ since } xyz = \lambda ax \text{ by FOC.}$$

(c) $\bar{H}\begin{bmatrix} \partial x^*/\partial a \\ \partial y^*/\partial a \\ \partial \lambda^*/\partial a \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ x \end{bmatrix}$

$$\frac{\partial x^*}{\partial a} = \frac{\begin{vmatrix} \lambda & 0.25x^{-0.5}y^{-0.5} & -a \\ 0 & -0.25x^{0.5}y^{-1.5} & -1 \\ x & -1 & 0 \end{vmatrix}}{|\bar{H}|} = \frac{\lambda(-1) + x\left(-0.25x^{-0.5}y^{-0.5} - 0.25ax^{0.5}y^{-1.5}\right)}{|\bar{H}|}$$

$$= \frac{(-)}{(+)} < 0 \quad \text{since } |\bar{H}| > 0 \text{ by SOC.}$$

$$\frac{\partial y^*}{\partial a} = \frac{\begin{vmatrix} -0.25x^{-1.5}y^{0.5} & \lambda & -a \\ 0.25x^{0.5}y^{0.5} & 0 & -1 \\ -a & x & 0 \end{vmatrix}}{|\bar{H}|} = \frac{(-a)\left(0.25x^{0.5}y^{-0.5}\right) + \left(-0.25x^{-0.5}y^{0.5} + a\lambda\right)}{|\bar{H}|}$$

$$= \frac{-\dfrac{a\lambda}{2} - \dfrac{a\lambda}{2} + a\lambda}{|\bar{H}|} = 0$$

since $\lambda = \dfrac{1}{2}\sqrt{\dfrac{x}{y}} = \dfrac{1}{2a}\sqrt{\dfrac{y}{x}}$ by FOC $\Rightarrow$ $0.25ax^{0.5}y^{-0.5} = 0.25x^{-0.5}y^{0.5} = \dfrac{a\lambda}{2}$

d) $$\bar{H}\begin{bmatrix}\partial x^*/\partial a\\ \partial y^*/\partial a\\ \partial z^*/\partial a\\ \partial \lambda^*/\partial a\end{bmatrix} = \begin{bmatrix}2x\lambda\\ 0\\ 0\\ x^2\end{bmatrix}$$

$$\frac{\partial x^*}{\partial a} = \frac{\begin{vmatrix}2x\lambda & 0 & 0 & -2ax\\ 0 & \dfrac{-A\beta}{y^2} - 2\lambda & 0 & -2y\\ 0 & 0 & \dfrac{-A\gamma}{z^2} - 2\lambda & -2z\\ x^2 & -2y & -2z & 0\end{vmatrix}}{|\bar{H}|}$$

$$= \frac{2x\lambda\begin{vmatrix}\dfrac{-A\beta}{y^2} - 2\lambda & 0 & -2y\\ 0 & \dfrac{-A\gamma}{z^2} - 2\lambda & -2z\\ -2y & -2z & 0\end{vmatrix} - x^2\begin{vmatrix}0 & 0 & -2ax\\ \dfrac{-A\beta}{y^2} - 2\lambda & 0 & -2y\\ 0 & \dfrac{-A\gamma}{z^2} - 2\lambda & -2z\end{vmatrix}}{|\bar{H}|}$$

$$= \frac{2x\lambda\left(-2y\left(-2y\left(\dfrac{A\gamma}{z^2} + 2\lambda\right)\right) + 2z\left(-2z\left(-\dfrac{A\beta}{y^2} - 2\lambda\right)\right)\right)}{|\bar{H}|}$$

$$- \frac{x^2\left(-2ax\left(-\dfrac{A\beta}{y^2} - 2\lambda\right)\left(-\dfrac{A\gamma}{z^2} - 2\lambda\right)\right)}{|\bar{H}|}$$

$$= \frac{(+)((-)[(-)(+)]) + (+)((-)[(-)-(+)])) - (+)[(-)((-)-(+))((-)-(+))]}{|\bar{H}|}$$

$$= \frac{(+)}{(-)} < 0 \quad \text{since } |\bar{H}| < 0 \text{ by SOC.}$$

$$\frac{\partial y^*}{\partial a} = \frac{\begin{vmatrix} \frac{-A\alpha}{x^2} - 2a\lambda & 2x\lambda & 0 & -2ax \\ 0 & 0 & 0 & -2y \\ 0 & 0 & \frac{-A\gamma}{z^2} - 2\lambda & -2z \\ -2ax & x^2 & -2z & 0 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-2y \begin{vmatrix} \frac{-A\alpha}{x^2} - 2a\lambda & 2x\lambda & 0 \\ 0 & 0 & \frac{-A\gamma}{z^2} - 2\lambda \\ -2ax & x^2 & -2z \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-2y\left(\frac{A\gamma}{z^2} + 2\lambda\right)\left(x^2\left(-\frac{A\alpha}{x^2} - 2a\lambda\right) + 4ax^2\lambda\right)}{|\bar{H}|}$$

$$= \frac{-2y\left(\frac{A\gamma}{z^2} + 2\lambda\right)\left(-A\alpha - 2ax^2\lambda + 4ax^2\lambda\right)}{|\bar{H}|} = 0 \quad \text{since } A\alpha = 2ax^2\lambda \quad \text{by FOC.}$$

(e) $\bar{H}\begin{bmatrix}\partial x^*/\partial a\\ \partial y^*/\partial a\\ \partial z^*/\partial a\\ \partial \lambda^*/\partial a\end{bmatrix} = \begin{bmatrix}\lambda\\0\\0\\x\end{bmatrix}$

$$\frac{\partial x^*}{\partial a} = \frac{\begin{vmatrix}\lambda & 0 & 0 & -a\\ 0 & 2 & 0 & -1\\ 0 & 0 & 2 & -1\\ x & -1 & -1 & 0\end{vmatrix}}{|\bar{H}|} = \frac{\lambda\begin{vmatrix}2 & 0 & -1\\ 0 & 2 & -1\\ -1 & -1 & 0\end{vmatrix} - x\begin{vmatrix}0 & 0 & -a\\ 2 & 0 & -1\\ 0 & 2 & -1\end{vmatrix}}{|\bar{H}|}$$

$$= \frac{\lambda(-2+(-2)) - x(-a(4))}{|\bar{H}|} = \frac{-4\lambda + 4ax}{|\bar{H}|}$$

the numerator of which cannot be signed even using the FOC.

$$\frac{\partial y^*}{\partial a} = \frac{\begin{vmatrix}2 & \lambda & 0 & -a\\ 0 & 0 & 0 & -1\\ 0 & 0 & 2 & -1\\ -a & x & -1 & 0\end{vmatrix}}{|\bar{H}|} = \frac{2\begin{vmatrix}0 & 0 & -1\\ 0 & 2 & -1\\ x & -1 & 0\end{vmatrix} + a\begin{vmatrix}\lambda & 0 & -a\\ 0 & 0 & -1\\ 0 & 2 & -1\end{vmatrix}}{|\bar{H}|}$$

$$= \frac{2(x(2)) + a((1)(2\lambda))}{|\bar{H}|} = \frac{4x + 2a\lambda}{|\bar{H}|} = \frac{(+)}{(-)} < 0 \quad \text{since } |\bar{H}| < 0 \text{ by SOC.}$$

(f) $\bar{H}\begin{bmatrix}\partial x^*/\partial a\\ \partial y^*/\partial a\\ \partial\lambda^*/\partial a\end{bmatrix} = \begin{bmatrix}-0.5x^{-0.5}y^{0.5}\\ -0.5x^{0.5}y^{-0.5}\\ -x^{0.5}y^{0.5}\end{bmatrix}$

$$\frac{\partial x^*}{\partial a} = \frac{\begin{vmatrix}-0.5x^{-0.5}y^{0.5} & -0.25a\lambda x^{-0.5}y^{-0.5} & -0.5ax^{-0.5}y^{0.5}\\ -0.5x^{0.5}y^{-0.5} & 0.25a\lambda x^{0.5}y^{-1.5} & -0.5ax^{0.5}y^{-0.5}\\ -x^{0.5}y^{0.5} & -0.5ax^{0.5}y^{-0.5} & 0\end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-x^{0.5}y^{0.5}[(-)(-)-(+)(-)] + 0.5ax^{0.5}y^{-0.5}(0.25a-0.25a)}{|\bar{H}|}$$

$$= \frac{(-)((+)-(-))}{(-)} > 0 \quad \text{since } |\bar{H}| < 0 \text{ by SOC.}$$

$$\frac{\partial y^*}{\partial a} = \frac{\begin{vmatrix}0.25a\lambda x^{-1.5}y^{0.5} & -0.5x^{-0.5}y^{0.5} & -0.5ax^{-0.5}y^{0.5}\\ -0.25a\lambda x^{-0.5}y^{-0.5} & -0.5x^{0.5}y^{-0.5} & -0.5ax^{0.5}y^{-0.5}\\ -0.5ax^{-0.5}y^{0.5} & -x^{0.5}y^{0.5} & 0\end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-0.5ax^{-0.5}y^{0.5}(0.25a-0.25a) + x^{0.5}y^{0.5}[(+)(-)-(-)(-)]}{|\bar{H}|}$$

$$= \frac{(+)((-)-(+))}{(-)} > 0$$

since $|\bar{H}| < 0$ by SOC.

(g) $$\bar{H}\begin{bmatrix} \partial x^*/\partial a \\ \partial y^*/\partial a \\ \partial z^*/\partial a \\ \partial \lambda^*/\partial a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial x^*}{\partial a} = \frac{\begin{vmatrix} 1 & 0 & 0 & -A\alpha/x \\ 0 & \lambda A\beta/y^2 & 0 & -A\beta/y \\ 0 & 0 & \lambda A\gamma/z^2 & -A\gamma/z \\ 0 & -A\beta/y & -A\gamma/z & 0 \end{vmatrix}}{|\bar{H}|} = \frac{\begin{vmatrix} \lambda A\beta/y^2 & 0 & -A\beta/y \\ 0 & \lambda A\gamma/z^2 & -A\gamma/z \\ -A\beta/y & -A\gamma/z & 0 \end{vmatrix}}{|\bar{H}|}$$

$= \dfrac{(-)}{(-)}$ since both the numerator and the denominator are border-preserving principal minors of $\bar{H}$

$$\frac{\partial y^*}{\partial a} = \frac{\begin{vmatrix} \lambda A\alpha/x^2 & 1 & 0 & -A\alpha/x \\ 0 & 0 & 0 & -A\beta/y \\ 0 & 0 & \lambda A\gamma/z^2 & -A\gamma/z \\ -A\alpha/x & 0 & -A\gamma/z & 0 \end{vmatrix}}{|\bar{H}|} = \frac{-\begin{vmatrix} 0 & 0 & -A\beta/y \\ 0 & \lambda A\gamma/z^2 & -A\gamma/z \\ -A\alpha/x & -A\gamma/z & 0 \end{vmatrix}}{|\bar{H}|}$$

$= \dfrac{(A\alpha/x)\left(\gamma A^2\beta\gamma/(yz^2)\right)}{|\bar{H}|} = \dfrac{(+)}{(-)} < 0$ since $|\bar{H}| < 0$ by SOC.

9.4

(d) $\bar{H}\begin{bmatrix} \partial x^* / \partial g_0 \\ \partial y^* / \partial g_0 \\ \partial z^* / \partial g_0 \\ \partial \lambda^* / \partial g_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$$\frac{\partial x^*}{\partial g_0} = \frac{\begin{vmatrix} 0 & 0 & 0 & -2ax \\ 0 & \frac{-A\beta}{y^2} - 2\lambda & 0 & -2y \\ 0 & 0 & \frac{-A\gamma}{z^2} - 2\lambda & -2z \\ -1 & -2y & -2z & 0 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{\begin{vmatrix} 0 & 0 & -2ax \\ \frac{-A\beta}{y^2} - 2\lambda & 0 & -2y \\ 0 & \frac{-A\gamma}{z^2} - 2\lambda & -2z \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-\left(-\frac{A\beta}{y^2} - 2\lambda \right)\left(0 - (-2ax)\left(\frac{-A\gamma}{z^2} - 2\lambda \right) \right)}{|\bar{H}|}$$

$$= \frac{-(-)[0-(-)(-)]}{(-)} > 0 \quad \text{since } |\bar{H}| < 0 \text{ by SOC.}$$

$$\frac{\partial y^*}{\partial g_0} = \frac{\begin{vmatrix} \frac{-A\alpha}{x^2} - 2a\lambda & 0 & 0 & -2ax \\ 0 & 0 & 0 & -2y \\ 0 & 0 & \frac{-A\gamma}{z^2} - 2\lambda & -2z \\ -2ax & -1 & -2z & 0 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-\begin{vmatrix} \frac{-A\alpha}{x^2} - 2a\lambda & 0 & -2ax \\ 0 & 0 & -2y \\ 0 & \frac{-A\gamma}{z^2} - 2\lambda & -2z \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{\left(\frac{A\alpha}{x^2} + 2a\lambda\right)\left(0 - (2y)\left(\frac{A\gamma}{z^2} + 2\lambda\right)\right)}{|\bar{H}|}$$

$$= \frac{(+)\,[\,0 - (+)(+)\,]}{(-)} > 0 \text{ since } |\bar{H}| < 0 \text{ by SOC.}$$

(e) $\bar{H}\begin{bmatrix} \partial x^*/\partial g_0 \\ \partial y^*/\partial g_0 \\ \partial z^*/\partial g_0 \\ \partial \lambda^*/\partial g_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$$\frac{\partial x^*}{\partial g_0} = \frac{\begin{vmatrix} 0 & 0 & 0 & -a \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 0 \end{vmatrix}}{|\bar{H}|} = \frac{\begin{vmatrix} 0 & 0 & -a \\ 2 & 0 & -1 \\ 0 & 2 & -1 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-4a}{|\bar{H}|} > 0 \text{ since } |\bar{H}| < 0 \text{ by the SOC.}$$

$$\frac{\partial y^*}{\partial g_0} = \frac{\begin{vmatrix} 2 & 0 & 0 & -a \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -a & -1 & -1 & 0 \end{vmatrix}}{|\bar{H}|} = \frac{-\begin{vmatrix} 2 & 0 & -a \\ 0 & 0 & -1 \\ 0 & 2 & -1 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-2(2)}{|\bar{H}|} > 0 \quad \text{since } |\bar{H}| < 0 \text{ by SOC.}$$

9.5

$$\mathfrak{L} = \sum_{i=1}^{n} p_i x_i + \lambda\left(U_0 - U(x_1,\ldots,x_n)\right)$$

FOC: $\mathfrak{L}_i = p_i - \lambda U_i = 0 \quad i = 1,\ldots,n$

$$\mathfrak{L}_\lambda = U_0 - U(x_1,\ldots,x_n) = 0$$

$$\bar{H} = \begin{bmatrix} -\lambda U_{11} & -\lambda U_{12} & \cdots & -\lambda U_{1n} & -U_1 \\ -\lambda U_{21} & -\lambda U_{22} & \cdots & -\lambda U_{2n} & -U_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda U_{n1} & -\lambda U_{n2} & \cdots & -\lambda U_{nn} & -U_n \\ -U_1 & -U_2 & \cdots & -U_n & 0 \end{bmatrix}$$

$$\bar{H} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \\ d\lambda \end{bmatrix} = \begin{bmatrix} -dp_1 \\ -dp_2 \\ \vdots \\ -dp_n \\ -dU_0 \end{bmatrix}$$

Now let all exogenous differentials equal 0 except dp_1:

$$\frac{\partial x_1^*}{\partial p_1} = \frac{dx_1}{dp_1}\Big|_{(dp_2 = \cdots = dp_n = dU_0 = 0)} = \frac{\begin{vmatrix} -1 & -\lambda U_{12} & \cdots & -\lambda U_{1n} & -U_1 \\ 0 & -\lambda U_{22} & \cdots & -\lambda U_{2n} & -U_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\lambda U_{n2} & \cdots & -\lambda U_{nn} & -U_n \\ 0 & -U_2 & \cdots & -U_n & 0 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{-\begin{vmatrix} -\lambda U_{22} & \cdots & -\lambda U_{2n} & -U_2 \\ \vdots & \ddots & \vdots & \vdots \\ -\lambda U_{n2} & \cdots & -\lambda U_{nn} & -U_n \\ -U_2 & \cdots & -U_n & 0 \end{vmatrix}}{|\bar{H}|}$$

The numerator is (-1) times a border-preserving principal minor of order n-1 of $\bar{H}$, while the numerator is a border-preserving principal minor of order n. By the SOC, the border-preserving principal minors alternate in sign, so the numerator and the denominator have the same signs. $\partial x_1^* / \partial p_1 > 0$.

Now letting all exogenous differentials equal zero except for dp_2,

$$\frac{\partial x_1^*}{\partial p_2} = \frac{dx_1}{dp_2}\Big|_{(dp_1 = dp_3 = \cdots = dp_n = dU_0 = 0)} = \frac{\begin{vmatrix} 0 & -\lambda U_{12} & \cdots & -\lambda U_{1n} & -U_1 \\ -1 & -\lambda U_{22} & \cdots & -\lambda U_{2n} & -U_2 \\ 0 & -\lambda U_{32} & \cdots & -\lambda U_{3n} & -U_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\lambda U_{n2} & \cdots & -\lambda U_{nn} & -U_n \\ 0 & -U_2 & \cdots & -U_n & 0 \end{vmatrix}}{|\bar{H}|}$$

$$= \frac{\begin{vmatrix} -\lambda U_{12} & \cdots & -\lambda U_{1n} & -U_1 \\ -\lambda U_{32} & \cdots & -\lambda U_{3n} & -U_3 \\ \vdots & \ddots & \vdots & \vdots \\ -\lambda U_{n2} & \cdots & -\lambda U_{nn} & -U_n \\ -U_2 & \cdots & -U_n & 0 \end{vmatrix}}{|\bar{H}|}$$

The numerator is not a border-preserving principal minor of $\bar{H}$, so it cannot be signed.