Due date: February 8, 2022 before 2.00 pm

Question 1 (60 Points)

Score....................

Consider the individual's portfolio choice problem given in the below equation:

$$\max_A E[U(\tilde{W})] = \max_A E[U(W_0(1 + r_f) + A(\tilde{r} - r_f))]$$

Assume the utility of this investor: $U(W) = ln(W)$ and the rate of return on the risky asset equals

$$\tilde{r} = \begin{cases} 4r_f & \text{with probability} \frac{1}{2} \\ -r_f & \text{with probability} \frac{1}{2} \end{cases}$$

Solve for the individual's proportion of initial wealth invested in the risky asset, $(\frac{A}{W_0})$.

$$\frac{dE[U(\tilde{W})]}{dA} = E[U'(\tilde{W})(\tilde{r}-r_f)] = 0$$

$$\bar{\tilde{r}} = \frac{1}{2}(4r_f) - \frac{1}{2}(r_f)$$

$$\frac{d^2E[U(\tilde{W})]}{d^2A} = E[U''(\tilde{W})(\tilde{r}-r_f)^2] \leq 0$$

$E[U'(\tilde{W})(\tilde{r}-r_f)] = 0 \rightarrow$ maximum

Note: $V = E[U(\tilde{W})] = \sum_{i=1}^{n} p_i U_i$ Diminishingly increasing in wealth (concave function)

$\rightarrow \frac{1}{2}U'(w^h)(r^h - r_f) + \frac{1}{2}U'(w^l)(r^l - r_f) = 0$

Note: $U(w) = \ln(w)$

$\therefore U'(w) = \frac{1}{w}$

sub $U'(w) = 1/w$ in equation

$\rightarrow \frac{1}{2} \cdot \frac{1}{w^h} \cdot (4r_f - r_f) + \frac{1}{2} \cdot \frac{1}{w^l}(-r_f - r_f) = 0$

$$\frac{3r_f}{2w^h} - \frac{r_f}{w^l} = 0$$

Note:

$W_0 \rightarrow \tilde{W}$

$w^h = W_0(1+r_f) + A(4r_f - r_f)$

$= W_0(1+r_f) + 3Ar_f$

$w^l = W_0(1+r_f) + A(-r_f - r_f)$

$= W_0(1+r_f) - 2Ar_f$

① $U'(w^h)(3r_f) > 0$

② $U'(w^l)(-2r_f) < 0$

$A^* > 0$

sub $w^h = W_0(1+r_f) + 3Ar_f$, $w^l = W_0(1+r_f) - 2Ar_f$

$$\rightarrow \frac{3r_f}{2[W_0(1+r_f) + 3Ar_f]} - \frac{r_f}{W_0(1+r_f) - 2Ar_f} = 0$$

$$\frac{3r_f}{2[W_0(1+r_f) + 3Ar_f]} = \frac{r_f}{W_0(1+r_f) - 2Ar_f}$$

$$\frac{3r_f}{2[W_0(1+r_f)+3Ar_f]} = \frac{r_f}{W_0(1+r_f)-2Ar_f}$$

$$3\cancel{r_f}\,[W_0(1+r_f)-2Ar_f] = 2\cancel{r_f}\,[W_0(1+r_f)+3Ar_f]$$

$$3W_0(1+r_f)-6Ar_f = 2W_0(1+r_f)+6Ar_f$$

$$W_0(1+r_f) = 12Ar_f$$

$$\frac{A}{W_0} = \frac{1+r_f}{12r_f}$$ → individual's proportion of ※ initial wealth invested in the risky asset

Question 2 (60 Points) Score.....................

An expected-utility-maximizing individual has constant relative-risk-aversion utility,

$$U(W) = \frac{W^\gamma}{\gamma}$$

,with relative-risk-aversion coefficient of $\gamma = -1$. The individual currently owns a product that has a probability p to failing,an event that would result in a loss of wealth that has a present value equal to L. With probability 1-p, the product will not fail and no loss will result. The individual is considering wherther to purchase an extended warranty on this product. The warranty costs C and would insure the individual against loss if the product fails. Assuming that the cost of the warranty exceeds the expected loss from the product's failure, determine the individual's level of wealth at which she would be just indifferent between purchasing or not purchasing the warranty.

Given : $\tilde{W}$

	output	probability
$\tilde{W}$	$W-L$	p
	W	$1-p$

$E[U(\tilde{W})] = \sum_{i=1}^{n} p_i U_i$, $U(W) = \frac{W^\gamma}{\gamma}$ with $\gamma = -1$

$\therefore U(W) = \frac{W^{-1}}{-1} = -\frac{1}{W}$

Therefore, $E[U(\tilde{W})] = p\left(-\frac{1}{W-L}\right) + (1-p)\left(-\frac{1}{W}\right)$

$= -\frac{p}{W-L} - \frac{1}{W} + \frac{p}{W}$

$= \frac{p-1}{W} - \frac{p}{W-L}$

To make she indifferent between purchasing or non-purchasing

Set, $U(W-C) = E[U(\tilde{W})]$

$-\frac{1}{W-C} = \frac{p-1}{W} - \frac{p}{W-L}$

$\frac{p}{W-L} = \frac{p-1}{W} + \frac{1}{W-C}$

$\frac{p}{W-L} = \frac{(p-1)(W-C)+W}{W^2-CW}$

$p(W^2-CW) = (W-L)[pW - pC - \cancel{W} + C + \cancel{W}]$

$\cancel{pW^2} - \cancel{pCW} = \cancel{pW^2} - pWL - \cancel{pWC} + LpC + WC - LC$

$0 = -pWL + LpC + WC - LC$

$pWL - WC = LpC - LC$

$W(pL - C) = LpC - LC$

$W = \frac{pLC - LC}{pL - C}$ ※ → individual's level of wealth

Question 3 (60 Points) Score.....................

Risk Aversion: Consider the following utility functions (Defined over wealth:W)

(1) U(W) $=-\frac{1}{W}$
(2) U(W) =ln(W)
(3) U(W) =-$W^{-\gamma}$
(4) U(W) =-exp(-γW) $= -e^{(-\gamma W)}$
(5) U(W) $=\frac{W^{\gamma}}{\gamma}$
(6) U(W) =αW-βW^2

Questions:

(a) Check that they are well behaved (U'>0 and U"<0) or state restriction on the parameters so that they are. For the utility function (6), take the positive α and β , and give the range of wealth over which the utility function is well behaved.

(b) Compute the absolute and relative risk aversion coefficients.

(c) What is the effect of parameter α (when relevant)?

(d) Classify the functions as increasing /decreasing risk aversion utility functions (both absolute and relative).

(a)

① $U(W) = -(W)^{-1}$

$U'(W) = \frac{1}{W^2} > 0$

$U''(W) = -\frac{2}{W^3} < 0$

② $U(W) = \ln(W)$

$U'(W) = \frac{1}{W} > 0$

$U''(W) = -\frac{1}{W^2} < 0$

③ $U(W) = -W^{-\gamma}$

$U'(W) = -(-\gamma)W^{-\gamma-1} = \frac{\gamma}{W^{\gamma+1}} > 0$ if $\gamma > 0$

$U''(W) = \frac{-\gamma^2-\gamma}{W^{\gamma+2}} < 0$ if $\gamma > 0$

④ $U(W) = -e^{(-\gamma W)}$

$U'(W) = -e^{(-\gamma W)} \cdot (-\gamma)$

$= \gamma e^{(-\gamma W)}$

$= \frac{\gamma}{e^{\gamma W}} > 0$ if $\gamma > 0$

$U''(W) = \gamma e^{(-\gamma W)} \cdot (-\gamma)$

$= -\gamma^2 e^{(-\gamma W)}$

$= \frac{-\gamma^2}{e^{\gamma W}} < 0$ if $\gamma > 0$

⑤ $U(W) = \frac{1}{\gamma} W^{\gamma}$

$U'(W) = \frac{1}{\cancel{\gamma}} \cdot \cancel{\gamma} W^{\gamma-1}$

$= W^{\gamma-1} > 0$

$U''(W) = (\gamma-1)W^{\gamma-2} < 0$ if $\gamma < 1$

⑥ $U(W) = \alpha W - \beta W^2$

$U'(W) = \alpha - 2\beta W > 0$ if $\alpha > 2\beta W$

$U''(W) = -2\beta < 0$, $\beta > 0$

(b)+(d) Absolute risk aversion (to decline in wealth) Relative risk aversion

$$\frac{\partial RW}{\partial W} = \frac{\partial -\frac{U''(W)}{U'(W)}}{\partial W} \qquad R_r(W) = WR(W)$$

① ARA: $R(W) = -\frac{-2/W^3}{1/W^2} = \frac{2}{\cancel{W^3}_{W}} \cdot \frac{\cancel{W^2}}{1} = \frac{2}{W}$

$\frac{\partial R(W)}{\partial W} = \frac{-2}{W^2} < 0 \rightarrow$ DARA = higher wealth less risk aversion

Decreasing absolute risk aversion

RRA: $R_r(W) = \frac{2}{W} \cdot W = 2 \rightarrow$ constant relative risk aversion

$\frac{dR_r(W)}{dW} = 0$

② ARA: $R(W) = -\frac{(-1/W^2)}{1/W} = \frac{1}{W}$

$\frac{\partial R(W)}{\partial W} = -\frac{1}{W^2} < 0 \rightarrow$ DARA Decreasing absolute risk aversion

RRA: $R_r(W) = \frac{1}{W} W = 1 \rightarrow$ constant relative risk aversion

$\frac{dR_r(W)}{dW} = 0$

③ ARA: $R(W) = -\frac{(-\gamma^2 - \gamma)W^{-\gamma-2}}{\cancel{\gamma}(W^{-\gamma-1})} = \frac{\gamma+1}{W}$

$\frac{\partial R(W)}{\partial W} = \frac{-\gamma-1}{W^2} < 0 \rightarrow$ DARA Decreasing absolute risk aversion

RRA: $R_r(W) = \frac{\gamma+1}{W} \cdot W$

$= \gamma + 1$

$\frac{dR_r(W)}{dW} = 0 \rightarrow$ constant relative risk aversion

④ ARA: $R(W) = \cancel{-}\frac{\cancel{-}\gamma^2 \cancel{e^{-\gamma W}}}{\gamma \cancel{e^{-\gamma W}}} = \gamma$

$\frac{\partial R(W)}{\partial W} = 0 \rightarrow$ CARA Constant absolute risk aversion

RRA: $R_r(W) = \gamma W$

$\frac{dR_r(W)}{dW} = \gamma > 0 \rightarrow$ Increase relative risk aversion

⑤ ARA: $R(W) = -\frac{(\gamma-1)W^{\gamma-2}}{W^{\gamma-1}} = \frac{-\gamma+1}{W}$

$\frac{\partial R(W)}{\partial W} = \frac{\gamma-1}{W^2} > 0 \rightarrow$ IARA increasing absolve risk aversion

RRA: $R_r(W) = \frac{-\gamma+1}{W} \cdot W$

$= -\gamma + 1$

$\frac{dR_r(W)}{dW} = 0 \rightarrow$ constant relative risk aversion

⑥ ARA: $R(W) = -\frac{-2\beta}{\alpha - 2\beta W} = \frac{2\beta}{\alpha - 2\beta W}$

$\frac{\partial R(W)}{\partial W} = 2\beta \frac{2\beta}{(\alpha - 2\beta W)^2} = \frac{4\beta^2}{(\alpha - 2\beta W)^2} > 0 \rightarrow$ IARA increasing absolve risk aversion

RRA: $R_r(W) = \frac{2\beta}{\alpha - 2\beta W} \cdot W = \frac{2\beta W}{\alpha - 2\beta W}$

$$\frac{dR_r(W)}{dW} = \frac{(\alpha - 2\beta W)2\beta - 2\beta W(-2\beta)}{(\alpha - 2\beta W)^2}$$

$= \frac{2\beta\alpha}{(\alpha - 2\beta W)^2} > 0 \rightarrow$ IRRA increasing relative risk aversion

(c) from ⑥ $U(W) = \alpha W - \beta W^2$

If α increase, $U'(W)$ will be larger (unwilling to pay large risk premium)
absolute risk aversion[R(W)] will be lower, and
relative risk aversion [$R_r(W)$] will also lower.

If α decrease, $U'(W)$ will be smaller
absolute risk aversion[R(W)] will be higher, and
relative risk aversion [$R_r(W)$] will also higher.