where s_k is an exogenous constant. Similarly, for human capital, the growth rate is

$$\dot{\hat{h}} = s_h \tilde{A} \hat{k}^{\alpha} \hat{h}^{\eta-1} - (\delta + n + x) = s_h \tilde{A} \cdot e^{\alpha \ln \hat{k}} \cdot e^{-(1-\eta)\ln \hat{h}} - (\delta + n + x) \tag{1.56}$$

where s_h is another exogenous constant. A shortcoming of this approach is that the rates of return to physical and human capital are not equated.

The growth rate of $\hat{y}$ is a weighted average of the growth rates of the two inputs:

$$\dot{\hat{y}}/\hat{y} = \alpha \cdot (\dot{\hat{k}}/\hat{k}) + \eta \cdot (\dot{\hat{h}}/\hat{h})$$

If we use equations (1.55) and (1.56) and take a two-dimensional first-order Taylor-series expansion, we get

$$\begin{aligned}\dot{\hat{y}}/\hat{y} = \big[&\alpha s_k \tilde{A} \cdot e^{-(1-\alpha)\ln \hat{k}^*} \cdot e^{\eta \ln \hat{h}^*} \cdot [-(1-\alpha)] \\ &+ \eta s_h \tilde{A} \cdot e^{\alpha \ln \hat{k}^*} \cdot e^{-(1-\eta)\ln \hat{h}^*} \cdot \alpha\big] \cdot (\ln \hat{k} - \ln \hat{k}^*) \\ &+ \big[\alpha s_k \tilde{A} \cdot e^{-(1-\alpha)\ln \hat{k}^*} \cdot e^{\eta \ln \hat{h}^*} \cdot \eta \\ &+ \eta s_h \tilde{A} \cdot e^{\alpha \ln \hat{k}} \cdot e^{-(1-\eta)\ln \hat{h}^*} \cdot [-(1-\eta)]\big] \cdot (\ln \hat{h} - \ln \hat{h}^*)\end{aligned}$$

The steady-state conditions derived from equations (1.55) and (1.56) can be used to get

$$\begin{aligned}\dot{\hat{y}}/\hat{y} &= -(1-\alpha-\eta) \cdot (\delta + n + x) \cdot [\alpha \cdot (\ln \hat{k} - \ln \hat{k}^*) + \eta \cdot (\ln \hat{h} - \ln \hat{h}^*)] \\ &= -\beta^* \cdot (\ln \hat{y} - \ln \hat{y}^*)\end{aligned} \tag{1.57}$$

Therefore, in the neighborhood of the steady state, the convergence coefficient is $\beta^* = (1-\alpha-\eta) \cdot (\delta + n + x)$, just as in equation (1.54).

1.3 Models of Endogenous Growth

1.3.1 Theoretical Dissatisfaction with Neoclassical Theory

In the mid-1980s it became increasingly clear that the standard neoclassical growth model was theoretically unsatisfactory as a tool to explore the determinants of long-run growth. We have seen that the model without technological change predicts that the economy will eventually converge to a steady state with zero per capita growth. The fundamental reason is the diminishing returns to capital. One way out of this problem was to broaden the concept of capital, notably to include human components, and then assume that diminishing returns did not apply to this broader class of capital. This approach is the one outlined in the next section and explored in detail in chapters 4 and 5. However, another view was that technological progress in the form of the generation of new ideas was the only way that an economy could escape from diminishing returns in the long run. Thus it became a priority to go beyond the treatment of technological progress as exogenous and, instead, to explain this progress within the model of growth. However, endogenous approaches to technological change encountered basic problems within the neoclassical model—the essential reason is the nonrival nature of the ideas that underlie technology.

Remember that a key characteristic of the state of technology, T, is that it is a nonrival input to the production process. Hence, the replication argument that we used before to justify the assumption of constant returns to scale suggests that the correct measure of scale is the two rival inputs, capital and labor. Hence, the concept of constant returns to scale that we used is homogeneity of degree one in K and L:

$$F(\lambda K, \lambda L, T) = \lambda \cdot F(K, L, T)$$

Recall also that Euler's theorem implies that a function that is homogeneous of degree one can be decomposed as

$$F(K, L, T) = F_K \cdot K + F_L \cdot L \tag{1.58}$$

In our analysis up to this point, we have been assuming that the same technology, T, is freely available to all firms. This availability is technically feasible because T is nonrival. However, it may be that T is at least partly excludable—for example, patent protection, secrecy, and experience might allow some producers to have access to technologies that are superior to those available to others. For the moment, we maintain the assumption that technology is nonexcludable, so that all producers have the same access. This assumption also means that a technological advance is immediately available to all producers.

We know from our previous analysis that perfectly competitive firms that take the input prices, R and w, as given end up equating the marginal products to the respective input prices, that is, $F_K = R$ and $F_L = w$. It follows from equation (1.58) that the factor payments exhaust the output, so that each firm's profit equals zero at every point in time.

Suppose that a firm has the option to pay a fixed cost, κ, to improve the technology from T to T'. Since the new technology would, by assumption, be freely available to all other producers, we know that the equilibrium values of R and w would again entail a zero flow of profit for each firm. Therefore, the firm that paid the fixed cost, κ, will end up losing money overall, because the fixed cost would not be recouped by positive profits at any future dates. It follows that the competitive, neoclassical model cannot sustain purposeful investment in technical change if technology is nonexcludable (as well as nonrival).

The obvious next step is to allow the technology to be at least partly excludable. To bring out the problems with this extension, consider the polar case of full excludability, that is, where each firm's technology is completely private. Assume, however, that there are infinitely many ways in which firms can improve knowledge from T to T' by paying the fixed cost κ—in other words, there is free entry into the business of creating formulas. Suppose that all firms begin with the technology T. Would an individual firm then have the incentive to pay κ to improve the technology to T'? In fact, the incentive appears to be enormous. At the existing input prices, R and w, a neoclassical firm with a superior technology would make a pure profit on each unit produced. Because of the assumed constant returns to scale, the firm would be motivated to hire all the capital and labor available in the economy. In this case, the firm would have lots of monopoly power and would likely no longer act as a perfect competitor in the goods and factor markets. So, the assumptions of the competitive model would break down.

A more basic problem with this result is that other firms would have perceived the same profit opportunity and would also have paid the cost κ to acquire the better technology, T'. However, when many firms improve their technology by the same amount, the competition pushes up the factor prices, R and w, so that the flow of profit is again zero. In this case, none of the firms can cover their fixed cost, κ, just as in the model in which technology was nonexcludable. Therefore, it is not an equilibrium for technological advance to occur (because all innovators make losses) and it is also not an equilibrium for this advance not to occur (because the potential profit to a single innovator is enormous).

These conceptual difficulties motivated researchers to introduce some aspects of imperfect competition to construct satisfactory models in which the level of the technology can be advanced by purposeful activity, such as R&D expenditures. This potential for endogenous technological progress and, hence, *endogenous growth*, may allow an escape from diminishing returns at the aggregate level. Models of this type were pioneered by Romer (1990) and Aghion and Howitt (1992); we consider them in chapters 6–8. For now, we deal only with models in which technology is either fixed or varying in an exogenous manner.

1.3.2 The *AK* Model

The key property of this class of endogenous-growth models is the absence of diminishing returns to capital. The simplest version of a production function without diminishing returns is the AK function:[29]

$$Y = AK \tag{1.59}$$

where A is a positive constant that reflects the level of the technology. The global absence of diminishing returns may seem unrealistic, but the idea becomes more plausible if we think of K in a broad sense to include human capital.[30] Output per capita is $y = Ak$, and the average and marginal products of capital are constant at the level $A > 0$.

29. We think that the first economist to use a production function of the AK type was von Neumann (1937).

30. Knight (1944) stressed the idea that diminishing returns might not apply to a broad concept of capital.

If we substitute $f(k)/k = A$ in equation (1.13), we get

$$\dot{k}/k = sA - (n + \delta)$$

We return here to the case of zero technological progress, $x = 0$, because we want to show that per capita growth can now occur in the long run even without exogenous technological change. For a graphical presentation, the main difference is that the downward-sloping saving curve, $s \cdot f(k)/k$, in figure 1.4 is replaced in figure 1.12 by the horizontal line at the level sA. The depreciation curve is still the same horizontal line at $n + \delta$. Hence, $\dot{k}/k$ is the vertical distance between the two lines, sA and $n + \delta$. We depict the case in which $sA > (n + \delta)$, so that $\dot{k}/k > 0$. Since the two lines are parallel, $\dot{k}/k$ is constant; in particular, it is independent of k. Therefore, k always grows at the steady-state rate, $(\dot{k}/k)^* = sA - (n + \delta)$.

Since $y = Ak$, $\dot{y}/y = \dot{k}/k$ at every point in time. In addition, since $c = (1 - s) \cdot y$, $\dot{c}/c = \dot{k}/k$ also applies. Hence, all the per capita variables in the model always grow at the same, constant rate, given by

$$\gamma^* = sA - (n + \delta) \tag{1.60}$$

Note that an economy described by the AK technology can display positive long-run per capita growth without any technological progress. Moreover, the per capita growth rate

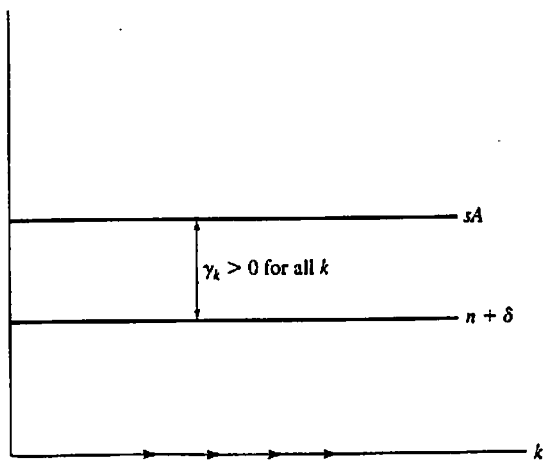

Figure 1.12
The AK Model. If the technology is AK, the saving curve, $s \cdot f(k)/k$, is a horizontal line at the level sA. If $sA > n + \delta$, perpetual growth of k occurs, even without technological progress.

shown in equation (1.60) depends on the behavioral parameters of the model, including s, A, and n. For example, unlike the neoclassical model, a higher saving rate, s, leads to a higher rate of long-run per capita growth, γ^*.[31] Similarly if the level of the technology, A, improves once and for all (or if the elimination of a governmental distortion effectively raises A), then the long-run growth rate is higher. Changes in the rates of depreciation, δ, and population growth, n, also have permanent effects on the per capita growth rate.

Unlike the neoclassical model, the AK formulation does not predict absolute or conditional convergence, that is, $\partial(\dot{y}/y)/\partial y = 0$ applies for all levels of y. Consider a group of economies that are structurally similar in that the parameters s, A, n, and δ are the same. The economies differ only in terms of their initial capital stocks per person, $k(0)$, and, hence, in $y(0)$ and $c(0)$. Since the model says that each economy grows at the same per capita rate, γ^*, regardless of its initial position, the prediction is that all the economies grow at the same per capita rate. This conclusion reflects the absence of diminishing returns. Another way to see this result is to observe that the AK model is just a Cobb–Douglas model with a unit capital share, $\alpha = 1$. The analysis of convergence in the previous section showed that the speed of convergence was given in equation (1.45) by $\beta^* = (1-\alpha)\cdot(x+n+\delta)$; hence, $\alpha = 1$ implies $\beta^* = 0$. This prediction is a substantial failing of the model, because conditional convergence appears to be an empirical regularity. See chapters 11 and 12 for a detailed discussion.

We mentioned that one way to think about the absence of diminishing returns to capital in the AK production function is to consider a broad concept of capital that encompassed physical and human components. In chapters 4 and 5 we consider in more detail models that allow for these two types of capital.

Other approaches have been used to eliminate the tendency for diminishing returns in the neoclassical model. We study in chapter 4 the notion of learning by doing, which was introduced by Arrow (1962) and used by Romer (1986). In these models, the experience with production or investment contributes to productivity. Moreover, the learning by one producer may raise the productivity of others through a process of spillovers of knowledge from one producer to another. Therefore, a larger economy-wide capital stock (or a greater cumulation of the aggregate of past production) improves the level of the technology for each producer. Consequently, diminishing returns to capital may not apply in the aggregate, and increasing returns are even possible. In a situation of increasing returns, each producer's average

31. With the AK production function, we can never get the kind of inefficient oversaving that is possible in the neoclassical model. A shift at some point in time to a permanently higher s means a lower level of c at that point but a permanently higher per capita growth rate, γ^*, and, hence, higher levels of c after some future date. This change cannot be described as inefficient because it may be desirable or undesirable depending on how households discount future levels of consumption.

product of capital, $f(k)/k$, tends to rise with the economy-wide value of k. Consequently, the $s \cdot f(k)/k$ curve in figure 1.4 tends to be upward sloping, at least over some range, and the growth rate, $\dot{k}/k$, rises with k in this range. Thus these kinds of models predict at least some intervals of per capita income in which economies tend to diverge. It is unclear, however, whether these divergence intervals are present in the data.

1.3.3 Endogenous Growth with Transitional Dynamics

The AK model delivers endogenous growth by avoiding diminishing returns to capital in the long run. This particular production function also implies, however, that the marginal and average products of capital are always constant and, hence, that growth rates do not exhibit the convergence property. It is possible to retain the feature of constant returns to capital in the long run, while restoring the convergence property—an idea brought out by Jones and Manuelli (1990).[32]

Consider again the expression for the growth rate of k from equation (1.13):

$$\dot{k}/k = s \cdot f(k)/k - (n + \delta) \tag{1.61}$$

If a steady state exists, the associated growth rate, $(\dot{k}/k)^*$, is constant by definition. A positive $(\dot{k}/k)^*$ means that k grows without bound. Equation (1.13) implies that it is necessary and sufficient for $(\dot{k}/k)^*$ to be positive to have the average product of capital, $f(k)/k$, remain above $(n+\delta)/s$ as k approaches infinity. In other words, if the average product approaches some limit, then $\lim_{k\to\infty}[f(k)/k] > (n+\delta)/s$ is necessary and sufficient for endogenous, steady-state growth.

If $f(k) \to \infty$ as $k \to \infty$, then an application of l'Hôpital's rule shows that the limits as k approaches infinity of the average product, $f(k)/k$, and the marginal product, $f'(k)$, are the same. (We assume here that $\lim_{k\to\infty}[f'(k)]$ exists.) Hence, the key condition for endogenous, steady-state growth is that $f'(k)$ be bounded sufficiently far above 0:

$$\lim_{k\to\infty} [f(k)/k] = \lim_{k\to\infty} [f'(k)] > (n+\delta)/s > 0$$

This inequality violates one of the standard Inada conditions in the neoclassical model, $\lim_{k\to\infty}[f'(k)] = 0$. Economically, the violation of this condition means that the tendency for diminishing returns to capital tends to disappear. In other words, the production function can exhibit diminishing or increasing returns to k when k is low, but the marginal product of capital must be bounded from below as k becomes large. A simple example, in which the production function converges asymptotically to the AK form, is

$$Y = F(K, L) = AK + BK^{\alpha}L^{1-\alpha} \tag{1.62}$$

32. See Kurz (1968) for a related discussion.

where $A > 0$, $B > 0$, and $0 < \alpha < 1$. Note that this production function is a combination of the AK and Cobb–Douglas functions. It exhibits constant returns to scale and positive and diminishing returns to labor and capital. However, one of the Inada conditions is violated because $\lim_{K\to\infty}(F_K) = A > 0$.

We can write the function in per capita terms as

$$y = f(k) = Ak + Bk^{\alpha}$$

The average product of capital is given by

$$f(k)/k = A + Bk^{-(1-\alpha)}$$

which is decreasing in k but approaches A as k tends to infinity.

The dynamics of this model can be analyzed with the usual expression from equation (1.13):

$$\dot{k}/k = s \cdot \left[A + Bk^{-(1-\alpha)}\right] - (n + \delta) \tag{1.63}$$

Figure 1.13 shows that the saving curve is downward sloping, and the line $n + \delta$ is horizontal. The difference from figure 1.4 is that, as k goes to infinity, the saving curve in figure 1.13 approaches the positive quantity sA, rather than 0. If $sA > n + \delta$, as assumed in the figure, the steady-state growth rate, $(\dot{k}/k)^*$, is positive.

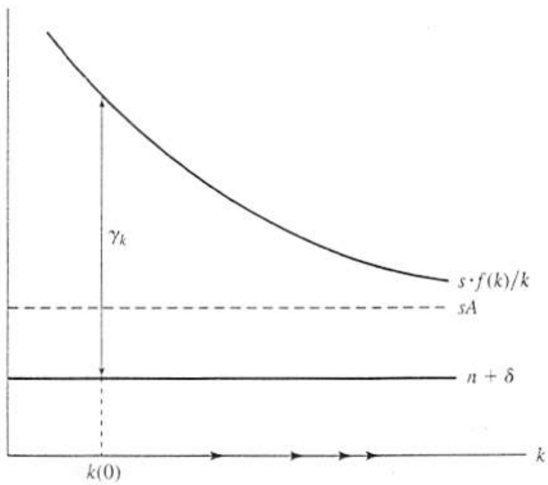

Figure 1.13
Endogenous growth with transitional dynamics. If the technology is $F(K, L) = AK + BK^{\alpha}L^{1-\alpha}$, the growth rate of k is diminishing for all k. If $sA > n + \delta$, the growth rate of k asymptotically approaches a positive constant, given by $sA - n - \delta$. Hence, endogenous growth coexists with a transition in which the growth rate diminishes as the economy develops.

This model yields endogenous, steady-state growth but also predicts conditional convergence, as in the neoclassical model. The reason is that the convergence property derives from the inverse relation between $f(k)/k$ and k, a relation that still holds in the model. Figure 1.13 shows that if two economies differ only in terms of their initial values, $k(0)$, the one with the smaller capital stock per person will grow faster in per capita terms.

1.3.4 Constant-Elasticity-of-Substitution Production Functions

Consider as another example the production function (due to Arrow et al., 1961) that has a constant elasticity of substitution (CES) between labor and capital:

$$Y = F(K, L) = A \cdot \{a \cdot (bK)^{\psi} + (1-a) \cdot [(1-b) \cdot L]^{\psi}\}^{1/\psi} \tag{1.64}$$

where $0 < a < 1$, $0 < b < 1$,[33] and $\psi < 1$. Note that the production function exhibits constant returns to scale for all values of ψ. The elasticity of substitution between capital and labor is $1/(1-\psi)$ (see the appendix, section 1.5.4). As $\psi \to -\infty$, the production function approaches a fixed-proportions technology (discussed in the next section), $Y = \min[bK, (1-b)L]$, where the elasticity of substitution is 0. As $\psi \to 0$, the production function approaches the Cobb–Douglas form, $Y = (\text{constant}) \cdot K^a L^{1-a}$, and the elasticity of substitution is 1 (see the appendix, section 1.5.4). For $\psi = 1$, the production function is linear, $Y = A \cdot [abK + (1-a) \cdot (1-b) \cdot L]$, so that K and L are perfect substitutes (infinite elasticity of substitution).

Divide both sides of equation (1.64) by L to get an expression for output per capita:

$$y = f(k) = A \cdot [a \cdot (bk)^{\psi} + (1-a) \cdot (1-b)^{\psi}]^{1/\psi}$$

The marginal and average products of capital are given, respectively, by

$$f'(k) = Aab^{\psi}[ab^{\psi} + (1-a) \cdot (1-b)^{\psi} \cdot k^{-\psi}]^{(1-\psi)/\psi}$$

$$f(k)/k = A[ab^{\psi} + (1-a) \cdot (1-b)^{\psi} \cdot k^{-\psi}]^{1/\psi}$$

Thus, $f'(k)$ and $f(k)/k$ are each positive and diminishing in k for all values of ψ.

We can study the dynamic behavior of a CES economy by returning to the expression from equation (1.13):

$$\dot{k}/k = s \cdot f(k)/k - (n + \delta) \tag{1.65}$$

33. The standard formulation does not include the terms b and $1-b$. The implication then is that the shares of K and L in total product each approach one-half as $\psi \to -\infty$. In our formulation, the shares of K and L approach b and $1-b$, respectively, as $\psi \to -\infty$.

If we graph versus k, then $s \cdot f(k)/k$ is a downward-sloping curve, $n + \delta$ is a horizontal line, and $\dot{k}/k$ is still represented by the vertical distance between the curve and the line. The behavior of the growth rate now depends, however, on the parameter ψ, which governs the elasticity of substitution between L and K.

Consider first the case $0 < \psi < 1$, that is, a high degree of substitution between L and K. The limits of the marginal and average products of capital in this case are

$$\lim_{k\to\infty} [f'(k)] = \lim_{k\to\infty} [f(k)/k] = Aba^{1/\psi} > 0$$

$$\lim_{k\to 0} [f'(k)] = \lim_{k\to 0} [f(k)/k] = \infty$$

Hence, the marginal and average products approach a positive constant, rather than 0, as k goes to infinity. In this sense, the CES production function with high substitution between the factors $(0 < \psi < 1)$ looks like the example in equation (1.62) in which diminishing returns vanished asymptotically. We therefore anticipate that this CES model can generate endogenous, steady-state growth.

Figure 1.14 shows the results graphically. The $s \cdot f(k)/k$ curve is downward sloping, and it asymptotes to the positive constant $sAb \cdot a^{1/\psi}$. If the saving rate is high enough, so that $sAb \cdot a^{1/\psi} > n + \delta$—as assumed in the figure—then the $s \cdot f(k)/k$ curve always lies above the $n + \delta$ line. In this case, the per capita growth rate is always positive, and the model

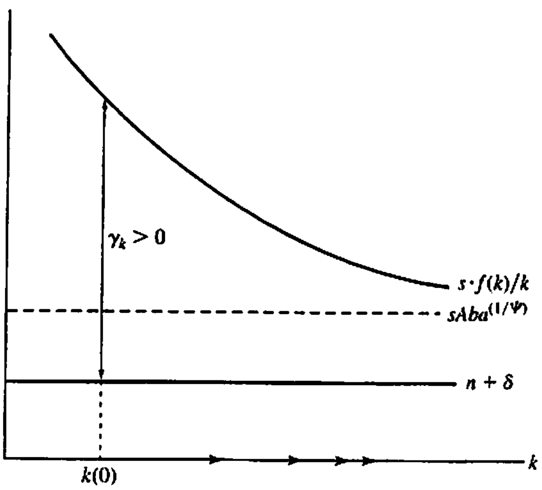

Figure 1.14
The CES model with $0 < \psi < 1$ and $sAb \cdot a^{1/\psi} > n + \delta$. If the CES technology exhibits a high elasticity of substitution $(0 < \psi < 1)$, endogenous growth arises if the parameters satisfy the inequality $sAb \cdot a^{1/\psi} > n + \delta$. Along the transition, the growth rate of k diminishes.

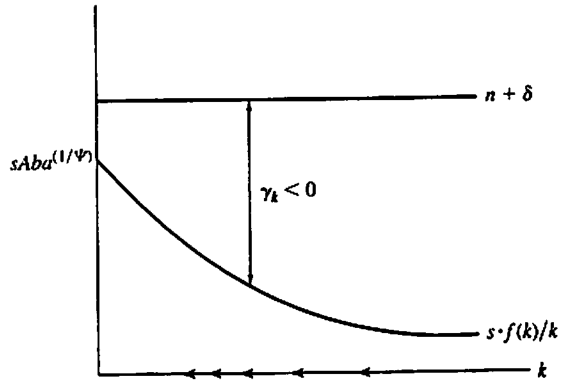

Figure 1.15
The CES model with $\psi < 0$ and $sAb \cdot a^{1/\psi} < n+\delta$. If the CES technology exhibits a low elasticity of substitution ($\psi < 0$), the growth rate of k would be negative for all levels of k if $sAb \cdot a^{1/\psi} < n+\delta$.

generates endogenous, steady-state growth at the rate

$$\gamma^* = sAb \cdot a^{1/\psi} - (n+\delta)$$

The dynamics of this model are similar to those described in figure 1.13.[34]

Assume now $\psi < 0$, that is, a low degree of substitution between L and K. The limits of the marginal and average products of capital in this case are

$$\lim_{k\to\infty} [f'(k)] = \lim_{k\to\infty} [f(k)/k] = 0$$
$$\lim_{k\to 0} [f'(k)] = \lim_{k\to 0} [f(k)/k] = Ab \cdot a^{1/\psi} < \infty$$

Since the marginal and average products approach 0 as k approaches infinity, the key Inada condition is satisfied, and the model does not generate endogenous growth. In this case, however, the violation of the Inada condition as k approaches 0 may cause problems. Suppose that the saving rate is low enough so that $sAb \cdot a^{1/\psi} < n+\delta$. In this case, the $s \cdot f(k)/k$ curve starts at a point below $n+\delta$, and it converges to 0 as k approaches infinity. Figure 1.15 shows, accordingly, that the curve never crosses the $n+\delta$ line, and, hence, no steady state exists with a positive value of k. Since the growth rate $\dot{k}/k$ is always negative, the economy shrinks over time, and k, y, and c all approach 0.[35]

34. If $0 < \psi < 1$ and $sAb \cdot a^{1/\psi} < n+\delta$, then the $s \cdot f(k)/k$ curve crosses $n+\delta$ at the steady-state value k^*, as in the standard neoclassical model of figure 1.4. Endogenous growth does not apply in this case.

35. If $\psi < 0$ and $sAb \cdot a^{1/\psi} > n+\delta$, then the $s \cdot f(k)/k$ curve again intersects the $n+\delta$ line at the steady-state value k^*.

Since the average product of capital, $f(k)/k$, is a negative function of k for all values of ψ, the growth rate $\dot{k}/k$ is also a negative function of k. The CES model therefore always exhibits the convergence property: for two economies with identical parameters and different initial values, $k(0)$, the one with the lower value of $k(0)$ has the higher value of $\dot{k}/k$. When the parameters differ across economies, the model predicts conditional convergence, as described before.

We can use the method developed earlier for the case of a Cobb–Douglas production function to derive a formula for the convergence coefficient in the neighborhood of the steady state. The result for a CES production function, which extends equation (1.45), is[36]

$$\beta^* = -(x+n+\delta)\cdot\left[1 - a\cdot\left(\frac{bsA}{x+n+\delta}\right)^{\psi}\right] \tag{1.66}$$

For the Cobb–Douglas case, where $\psi = 0$ and $a = \alpha$, equation (1.66) reduces to equation (1.45). For $\psi \neq 0$, a new result is that β^* in equation (1.66) depends on s and A. If $\psi > 0$ (high substitutability between L and K), then β^* falls with sA, and vice versa if $\psi < 0$. The coefficient β^* is independent of s and A only in the Cobb–Douglas case, where $\psi = 0$.

1.4 Other Production Functions . . . Other Growth Theories

1.4.1 The Leontief Production Function and the Harrod–Domar Controversy

A production function that was used prior to the neoclassical one is the Leontief (1941), or fixed-proportions, function,

$$Y = F(K, L) = \min(AK, BL) \tag{1.67}$$

where $A > 0$ and $B > 0$ are constants. This specification, which corresponds to $\psi \to -\infty$ in the CES form in equation (1.64), was used by Harrod (1939) and Domar (1946). With fixed proportions, if the available capital stock and labor force happen to be such that $AK = BL$, then all workers and machines are fully employed. If K and L are such that $AK > BL$, then only the quantity of capital $(B/A)\cdot L$ is used, and the remainder remains idle. Conversely, if $AK < BL$, then only the amount of labor $(A/B)\cdot K$ is used, and the remainder is unemployed. The assumption of no substitution between capital and labor led Harrod and Domar to predict that capitalist economies would have undesirable outcomes in the form of perpetual increases in unemployed workers or machines. We provide here a brief analysis of the Harrod–Domar model using the tools developed earlier in this chapter.

36. See Chua (1993) for additional discussion. The formula for β in equation (1.66) applies only for cases in which the steady-state level k^* exists. If $0 < \psi < 1$, it applies for $bsA\cdot a^{1/\psi} < x+n+\delta$. If $\psi < 0$, it applies for $bsA\cdot a^{1/\psi} > x+n+\delta$.