

Assignment 3

1. (a) Prove the statement:
 “There is a pair of real numbers x and y such that $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. ”

 (b) Disprove the statement: “For all real numbers x and y , $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. ”
2. Show that “for any integer n , if $n^3 + 5$ is odd, then n is even,” by using
 - a) a proof by contraposition,
 - b) a proof by contradiction.
3. Prove by the **method of exhaustion** that “ $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$.”
4. Use the **proof by cases** to show that “ for any integer n , $n^2 \geq n$.”
 [Hint: Consider 3 cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$]
5. Consider the statement: for $n \geq 1$,

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

Suppose we want to prove the above statement by **mathematical induction**.

- (a) What is $P(n)$?
- (b) Write $P(1)$: Is $P(1)$ true?
- (c) Write $P(k)$:
- (d) Write $P(k + 1)$:
- (e) Prove the above statement: $\sum_{j=1}^n \frac{1}{2^j} = \frac{2^n - 1}{2^n}$, by using **mathematical induction**.
6. Use mathematical induction proof to show that

$$n! < n^n,$$
 for any integer n that is greater than 1.
7. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
8. (Optional) Use the method of constructive proof to show that:
 if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.
9. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.
10. (Optional) A sequence $a_1, a_2, \dots$ is defined recursively by

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for } i \geq 2.$$

Show that

$$a_n = 3 \cdot 7^{n-1} \quad \text{for } n \geq 1.$$