Differentiation

Techniques of Differentiation

If it were always necessary to compute derivatives directly from the definition, as we did in the previous section, such computations would be tedious and the evaluation of some limits would require ingenuity. Fortunately, several rules have been developed for finding derivatives without having to use definition directly.

Derivative of a Constant Function: For any constant *c*

$$\frac{d}{dx}(c) = 0$$

Power Function:

$$\frac{d}{dx}(x) = 1$$

The Power Rule: If *n* is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Constant Multiple Rule: If *c* is a constant and *f* is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

The Sum Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Product Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The Quotient Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

- Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is *x* and the dependent variable is *y*, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols *D* and *d/dx* are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating a derivative.

Example :

(a) If $f(x) = x^5$. Find $f'(x)$.

(b) If $y = m^7$. Find $\dfrac{dy}{dm}$.

(c) If $y = x^{100}$. Find y'.

(d) Find $\dfrac{d}{dx}\left(2x^6 - x^4 + 3x^3 - 5\right)$

(e) Find y' where $y = \left(2t^2\right)\left(t^4 - 10\right)$.

(f) Find y' where $y = \left(x^{-3} + 1\right)\left(x^2 - x\right)$.

(g) Let $y = \dfrac{4x^5 - 2x^3 + 1}{x^2 + 1}$. Find y'.

(h) Let $y = \dfrac{\sqrt{x}}{4x^2 + 1}$. Find y'

The Chain Rule

If *f* and *g* are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then *F* is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

<u>i.e.</u> If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- The Power Rule Combined with the Chain Rule:

If *n* is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$$

Example: Find the derivative of the function.

(1) $f(x) = \sqrt{x^2 + 1}$

(2) $y = \dfrac{1}{\sqrt[3]{x^2 + 1}}$

(3) $f(x) = \left(\dfrac{x-2}{2x+1}\right)^6$

Derivatives of Logarithmic and Exponential Functions

- Derivatives of Exponential Functions

Theorem: For any constant $a > 0$

$$\frac{d}{dx}(a^x) = a^x lna$$

Theorem: Derivative of the Natural Exponential Function

$$\frac{d}{dx}e^x = e^x$$

Example: Differentiate the function

$$f(x) = \frac{e^{-4x}}{x+1}$$

- Derivative of Logarithmic Functions

Recall:

$$log_a x = \frac{lnx}{lna}$$

Since lna is a constant, we can differentiate as follows:

$$\frac{d}{dx}(log_a x) = \frac{d}{dx}\frac{lnx}{lna} = \frac{1}{lna}\frac{d}{dx}(lnx) = \frac{1}{xlna}$$

Hence, for any constant $a > 0$

$$\frac{d}{dx}(log_a x) = \frac{1}{xlna}$$

Since $lne = 1$ and $log_e x = lnx$, and if we put $a = e$, then we have

$$\frac{d}{dx}(lnx) = \frac{d}{dx}(log_e x) = \frac{1}{xlne} = \frac{1}{x}$$

Theorem:

$$\frac{d}{dx}(lnx) = \frac{1}{x}$$

In general, if we combine the formula above with the Chain Rule, we get

$$\frac{d}{dx}[lng(x)] = \frac{g'(x)}{g(x)}$$

Example: Differentiate the function :

(a) $y = ln\frac{x+1}{\sqrt{x-2}}$

(b) $y = \left(e^{-x}\right) log_2\left(\frac{x}{3^x}\right)$

Higher Derivatives

If we take the derivative of $y = f(x)$, we get another function $f'(x)$ or dy/dx.

We can also differentiate that function, the result is called the *second derivative*, denoted as

$$f''(x) \quad OR \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad OR \quad D^2 f(x)$$

Hence, the third derivative $f'''(x)$ is the derivative of the second derivative:

$$f''' = \left(f''\right)'$$

So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

Notation:

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3 f(x)$$

In general, we write

$$f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

as the nth derivative of $f(x)$.

Example: Find the second derivative of the function.

$(a)\ f(x) = x^3 + 2x^2 - 1$

$(b)\ y = \dfrac{x^2+1}{\sqrt{x}}$