Solution: Exercise 3 (Part 3)

1. (a) Prove the statement:
   " For all integer n, if n is odd, then $5n + 4$ is odd. "

   (b) Disprove the statement:
   " For all integers x and y, if $3x - 5y$ is even, then x and y are both even. "

   (c) Prove or disprove the statement:
   " For all real numbers x and y, if x and y are rational, then x^y is rational. "

   Solution:

   (a) The statement:
   " For all integer n, if n is odd, then $5n + 4$ is odd. "
   is a conditional statement with universal quantifier. So, we can prove this by using the **direct proof**.
   Suppose n is odd. Then, by definition, we can write $n = 2a + 1$, for some integer $a \in \mathbb{Z}$. We can show that $5n + 4$ is odd as follows.

   $$5n + 4 = 5(2a + 1) + 4 = 10a + 5 + 4 = 10a + 9 = 2(5a + 4) + 1.$$

   That is, we can write $5n + 4 = 2k + 1$, where $k = 5a + 4$ is an integer (since a is an integer). Therefore, $5n + 4$ is odd. ■

   (b) The statement" For all integers x and y, if $3x - 5y$ is even, then x and y are both even. " is a universal statement. We can **disprove** this statement by finding a counterexample that make this statement false or make its **negation** true. The negation is given by

   " there exist integers x and y, such that $3x - 5y$ is even, and (but) x or y is odd (possible to have both x and y odd). "

   Let $x = 1 \in \mathbb{Z}$ and $y = -1 \in \mathbb{Z}$. Notice that x and y are odd and $3x - 5y = 3(1) - 5(-1) = 8$ which is even. Hence, a counterexample is $x = 1$, $y = -1$ and we disprove the given statement.

   Note: There are many other possible counterexamples. ■

   (c) Notice that $x = 2$, $y = \frac{1}{2}$ are rational numbers. However $x^y = \sqrt{2}$ is not rational (i.e. irrational). That is, we found a counterexample that makes its negation true. In particular,

   The statement"" For all real numbers x and y, if x and y are rational, then x^y is rational. " is a universal statement. We can **disprove** this statement by finding a counterexample that make this statement false or make its **negation** true. The negation is given by

   " there exist integers x and y, such that x and y are rational, and (but) x^y is irrational. "

   One of the counterexamples is $x = 2, y = 1/2$.

   Note: There are many other possible counterexamples. ■

2. Show that "for all integer n, if $5n - 1$ is even, then n is odd." by using
   a) a proof by contraposition,
   b) a proof by contradiction.

Solution:

(a)**Proof by contraposition:**

The contrapositive of the given statement is

"for all integer n, if n is even (not odd), then $5n - 1$ is odd (not even)."

Let $n \in \mathbb{Z}$. Suppose n is even. Then we can write
$n = 2k$, for some $k \in \mathbb{Z}$ and
$5n - 1 = 5(2k) - 1 = 2(5k - 1) + 1$.
That is, $5n - 1 = 2r + 1$ for some $r = 5k - 1 \in \mathbb{Z}$. We can conclude that $5n - 1$ is odd.

The contrapositive is equivalent to the original statement and therefore showing that its contrapositive is true implies that the original statement is also true. ■

(b) **Proof by contradiction:**

Suppose not. I.e. suppose that its negation is true:

"there exist an integer n, such that $5n - 1$ is even, and n is even."

Then since n is even, we can write $n = 2k$, for some $k \in \mathbb{Z}$. Consider $5n - 1$:

$$5n - 1 = 5(2k) - 1 = 2\underbrace{(5k - 1)}_{=:r \in \mathbb{Z}} + 1 = 2r + 1,$$

which is an odd integer, by definition, since $r = 5k - 1$ is an integer.

This contradicts to the assumption that $5n - 1$ is even. Hence its negation is false and the statement itself is true. ■

3. Prove or disprove that " $(n + 1)^3 \geq 3^n$ for any positive integer n that is less than 5."

   Solution: Given that n is any positive integer that is less than 5, we will consider $n = 1, 2, 3,$ or 4.

   For $n = 1$, $(1 + 1)^3 = 8$ and $3^1 = 3$, so $(n + 1)^3 \geq 3^n$ is true.
   For $n = 2$, $(2 + 1)^3 = 27$ and $3^2 = 9$, so $(n + 1)^3 \geq 3^n$ is true.
   For $n = 3$, $(3 + 1)^3 = 64$ and $3^3 = 27$, so $(n + 1)^3 \geq 3^n$ is true.
   For $n = 4$, $(4 + 1)^3 = 125$ and $3^4 = 81$, so $(n + 1)^3 \geq 3^n$ is true.

   Therefore, we have used **method of exhaustion** to show that $(n + 1)^3 \geq 3^n$ for any positive integer n with $0 < n < 5$. ■

4. Use the **proof by cases** to show that " Prove that for all integers m and n, $m + n$ and $m - n$ are either both odd or both even."

   [Hint: Consider 4 cases of even and odd for m and n]

   Solution: We will consider 4 cases for m and n.

- Case I: m is even and n is even.
That is, $m = 2k$ and $n = 2s$ for some integers $k, s \in \mathbb{Z}$. which implies $m - n$ is even because $k - s$ is an integer.
$$m + n = 2k + 2s = 2(k + s)$$
Then, which implies $m + n$ is even because $k + s$ is an integer. Also,
$$m - n = 2k - 2s = 2(k - s)$$
Hence, both $m + n$ and $m - n$ are both even in this case.
- Case II: m is even and n is odd.
That is, $m = 2k$ and $n = 2a + 1$ for some integers $k, s \in \mathbb{Z}$. Then,
$$m + n = 2k + (2a + 1) = 2(k + a) + 1$$
which implies $m + n$ is odd because $k + a$ is an integer. Also,
$$m - n = 2k - (2a + 1) = 2(k - a) - 1 = 2(k - a) - 1 + 2 - 2 = 2(k - a - 1) + 1$$
which implies $m - n$ is odd because $k - a - 1$ is an integer. Hence, both $m + n$ and $m - n$ are both odd in this case.
- Case III: m is odd and n is even. That is, $m = 2b + 1$ and $n = 2s$ for some integers $k, s \in \mathbb{Z}$. Then,
$$m + n = (2b + 1) + 2s = 2(b + s) + 1$$
which implies $m + n$ is odd because $b + s$ is an integer. Also,
$$m - n = (2b + 1) - 2s = 2(b - s) + 1$$
which implies $m - n$ is odd because $b - s$ is an integer. Hence, both $m + n$ and $m - n$ are both odd in this case.
- Case IV: m is odd and n is odd.
That is, $m = 2b + 1$ and $n = 2a + 1$ for some integers $k, s \in \mathbb{Z}$. Then,
$$m + n = (2b + 1) + (2a + 1) = 2(b + a + 1)$$
which implies $m + n$ is even, by definition, because $b + a + 1$ is an integer. Also,
$$m - n = (2b + 1) - (2a + 1) = 2(b - a)$$
which implies $m - n$ is odd, by definition, because $b - a$ is an integer. Hence, both $m + n$ and $m - n$ are both even in this case. ■

5. Let $P(n)$ be the statement " If $n > 1$, then $n^2 > n + 1$ " with the domain of n consisting of all integers. Is the statement $P(1)$ true? Explain your answer.

Solution: This statement is true by **vacuous proof**. $P(1)$ is *If* $1 > 1$, *then* $1^2 > 2$. Since the hypothesis $1 > 1$ is false. This tells us that $P(1)$ is automatically true. ■

6. Consider the statement: for any integer $n \geq 2$,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < \frac{2n-1}{n}.$$

Suppose we want to prove the above statement by **mathematical induction**.

(a) What is $P(n)$?

(b) Write $P(2)$: Is $P(2)$ true?

(c) Write $P(k)$:

(d) Write $P(k+1)$:

(e) Prove the above statement: $\sum_{j=1}^{n} j^{-2} = \frac{2n-1}{n}$ by using **mathematical induction**.

Solution:

(a) What is $P(n)$?
We will first simplify

$$\frac{2n-1}{n} = 2 - \frac{1}{n}.$$

$P(n)$ is the statement

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < \ 2 - \frac{1}{n}.$$

(b) Write $P(2)$: Is $P(2)$ true?
$P(0) : 1 + \frac{1}{4} < 2 - \frac{1}{2}$. is true since the left-hand side is $1 + \frac{1}{4} = \frac{5}{4} = 1.25$ and the right-hand side is $2 - \frac{1}{2} = \frac{3}{2} = 1.5$.

(c) Write $P(k)$:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} < \ 2 - \frac{1}{k}.$$

(d) Write $P(k+1)$:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \ 2 - \frac{1}{k+1}.$$

(e) Prove the above statement: $\sum_{j=1}^{n} j^{-2} = \frac{2n-1}{n}$ by using **mathematical induction**.
We want to prove that $P(n)$ is true for all integer $n \geq 2$.
(I) **Basis step:** $P(2) : 1 + \frac{1}{4} < 2 - \frac{1}{2}$ gives $1.25 < 1.5$ (from $1 + \frac{1}{4} = \frac{5}{4} = 1.25$ and $2 - \frac{1}{2} = \frac{3}{2} = 1.5$), which is true.
(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 2$.
Assume that $P(k)$ is true:
$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} < \ 2 - \frac{1}{k}$. ————(★) "inductive hypothesis"
We want to show that $P(k+1)$ is true:
$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \ 2 - \frac{1}{k+1}$.

Consider the left-hand side of $P(k+1)$.

$$\underbrace{1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}}_{(\bigstar)}+\frac{1}{(k+1)^2} \quad < \quad 2-\frac{1}{k}+\frac{1}{(k+1)^2} \text{ by } (\bigstar)$$

Notice that we want to have the right-hand side $2-\frac{1}{k+1}$ as in $P(k+1)$, however, we now have $2-\frac{1}{k}+\frac{1}{(k+1)^2}$, so we **have to justify** that (*)-$\boxed{2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}}$, or $\boxed{\frac{1}{k}-\frac{1}{(k+1)^2}>\frac{1}{k+1}}$ which can be done as follows.

Notice that

$$\frac{1}{k}-\frac{1}{(k+1)^2}=\frac{(k+1)^2-k}{k}(k+1)^2=\frac{k^2+k+1}{k(k+1)^2}$$

and (*) is therefore equivalent to $\frac{1}{k+1}<\frac{k^2+k+1}{k(k+1)^2}$ or $k(k+1)<k^2+k+1$ or $\boxed{k^2+k<k^2+k+1}$, which is always true for $k\geq 2$. From above, the formal steps are given by: for $k\geq 2$,

$$\begin{aligned}
k^2+k &< k^2+k+1\\
1 &< \frac{k^2+k+1}{k(k+1)^2}=\frac{k+1}{k}-\frac{1}{k+1}\\
\text{[Dividing by } k+1\text{:]}\qquad \frac{1}{k+1} &< \frac{1}{k}-\frac{1}{(k+1)^2}\\
-\frac{1}{k}+\frac{1}{(k+1)^2} &< -\frac{1}{k+1}\\
\text{[Adding 2:]}\qquad 2-\frac{1}{k}+\frac{1}{(k+1)^2} &< 2-\frac{1}{k+1}-----------(\blacklozenge)
\end{aligned}$$

We can go back to consider the left-hand side of $P(k+1)$:

$$\begin{aligned}
\underbrace{1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}}_{(\bigstar)}+\frac{1}{(k+1)^2} &< 2-\frac{1}{k}+\frac{1}{(k+1)^2} \qquad \text{by } (\bigstar)\\
&< 2-\frac{1}{k+1} \qquad \text{by } (\blacklozenge)
\end{aligned}$$

which implies that $P(k+1)$ is true. That is, $P(k)$ implies $P(k+1)$. Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for all $n\geq 2$ by the induction proof. ■

7. Use mathematical induction proof to show that

$$2^n>n^2$$

for an integer greater than 4.

Solution:

Let $P(n)$ be the statement $2^n>n^2$.

We want to prove that $P(n)$ is true for all integer $n \geq 5$.

(I) **Basis step:** $P(5)$: $2^5 > n^2$ or $32 > 25$, which is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 5$.
Assume that $P(k)$ is true:
$P(k)$: $2^k > k^2$. ————(★) "inductive hypothesis"
We want to show that $P(k+1)$ is true: $2^{k+1} > (k+1)^2$.

Note that for $k \geq 5$, we have $(k-2) \geq 3$ and $k(k-2) \geq 15 > 1$:

$$\begin{aligned} k(k-2) &> 1 \\ k^2 - 2k &> 1 \\ k^2 + k^2 - 2k &> k^2 + 1 \\ 2k^2 &> k^2 + 2k + 1 = (k+1)^2. \end{aligned}$$

That is, $\boxed{2k^2 > (k+1)^2 \text{ for any integer } k \geq 5}$. ————————-(♠)

Consider 2^{k+1} from $P(k+1)$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 && \text{——————by "inductive hypothesis" (★) } 2^k > k^2 \\ &> (k+1)^2 && \text{——————by (♠) } 2k^2 > (k+1)^2 \end{aligned}$$

which implies that $P(k+1)$ is true.

Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 0$ by the induction proof. ■

Remark: To show (♠), it is also possible to use the proof by induction (see next page).

Note that to show (♠), it is also possible to use the proof by induction. Let $Q(n)$ be the statement $\boxed{2n^2 > (n+1)^2}$.

We want to prove that $Q(n)$ is true for all integer $n \geq 5$.
(i) **Basis step:** $Q(5) : 2 \cdot 5^2 > (5+1)^2$ or $50 > 36$, which is true.
(ii) **Inductive step:** Assume that $Q(k)$: $2k^2 > (k+1)^2$——(★★).
We want to show that $Q(k+1)$: $2(k+1)^2 > (k+2)^2$.

$$\begin{aligned}
2(k+1)^2 &= 2k^2 + 4k + 2 \\
&> (k+1)^2 + 4k + 2 \qquad \text{by “inductive hypothesis” (★★) } 2^k > k^2 \\
&= k^2 + 2k + 1 + 4k + 2 \\
&= (k^2 + 4k + 4) + \underbrace{2k - 1} \\
&= (k+2)^2 + \underbrace{2k-1}_{\geq 0} \\
&> (k+2)^2 \qquad \text{since } 2k - 1 > 0 \text{ for } k \geq 5
\end{aligned}$$

which implies that $2(k+1)^2 > (k+2)^2$ and $P(k+1)$ is true.
From (i) and (ii), $\boxed{2n^2 > (n+1)^2}$ for all $n \geq 5$.

8. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

   Solution: Contradiction proof

9. (Optional) Use the method of constructive proof to show that:
   if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

   Solution: Constructive proof
   Let $r, s \in \mathbb{R}$ such that $r < s$. Let
   $$x = \frac{r+s}{2}.$$
   We will show that for this particular x has the value between the r and s.

$$\begin{aligned}
r &< s \\
r + r &< s + r \\
\underbrace{\frac{r+r}{2}}_{=r} &< \underbrace{\frac{s+r}{2}}_{=x} \quad \Rightarrow \quad r < x
\end{aligned}$$

$$\begin{aligned}
r &< s \\
r + s &< s + s \\
\underbrace{\frac{r+s}{2}}_{=x} &< \underbrace{\frac{s+s}{2}}_{=s} \quad \Rightarrow \quad x < s
\end{aligned}$$

That is, for any given r and s, we can always find $x = \frac{r+s}{2}$ such that $r < x < s$.

Note that it is also possible to use a different value of x. ■

10. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

    Solution: Let r be any rational number and s be any irrational number. We want to show that $r - s$ is irrational.

    To prove this by contradiction, we will suppose that $r - s$ is rational. Then we can write $r = \frac{a}{b}$ and $r - s = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$. That is,

    $$\frac{a}{b} - s = \frac{c}{d}$$

    and so

    $$s = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

    which implies that s is rational. This is a contradiction to the assumption that s is irrational. Therefore, the the given statement is true by contradiction proof. ■

11. (Optional) Show that at least four of any 22 days must fall on the same day of the week. [Hint: Use contradiction proof.]

    Solution: Let p be the statement
    At least four of 22 chosen days fall on the same day of the week.
    Suppose that ∼p is true. This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts the premise that we have 22 days under consideration.

    That is, if r is the statement that 22 days are chosen, then we have shown that $\sim p \to (r \wedge \sim r)$, which means there is a **contradiction**. Consequently, we know that p is true.We have proved that at least four of 22 chosen days fall on the same day of the week. ■

12. (Optional) Show that these statements about the integer n are equivalent:
    p_1: n is even. p_2: $n - 1$ is odd. p_3: n^2 is even.

    [Hint: We will show that these three statements are equivalent by showing that the conditional statements $p_1 \to p_2$, $p_2 \to p_3$, and $p_3 \to p_1$ are true.]

    Solution:

13. (Optional) A sequence $a_1, a_2, \ldots$ is defined recursively by

    $$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$

    Show that

    $$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$

Solution:

Proof by mathematical induction:

From the given definition :

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2. \qquad ----------------\circledast$$

Let $P(n)$ be the statement $a_n = 3 \cdot 7^{n-1}$.
We want to prove that $P(n)$ is true for all integer $n \geq 1$.

(I) **Basis step:** Show that $P(1)$ is true.
$P(1)$: $a_1 = 3 \cdot 7^{1-1}$.
Since $a_1 = 3 \cdot 7^{1-1} = 3 \cdot 7^0 = 3 \cdot 1 = 3$, which is the same as $a_1 = 3$ from the definition $\circledast$.
Hence $P(1)$ is true.

(II)**Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.

Assume that $P(k) : a_k = 3 \cdot 7^{k-1}$ is true. ————————($\bigstar$) "inductive hypothesis"
We want to show that $P(k+1) : a_{k+1} = 3 \cdot 7^{(k+1)-1}$, or $a_{k+1} = 3 \cdot 7^k$ is true. Consider from the definition $\circledast$

$$\begin{aligned} a_{k+1} &= 7a_{[(k+1)-1]} \\ &= 7\ a_k \\ &= 7\ [3 \cdot 7^{k-1}] && \text{by } (\bigstar) \text{ "inductive hypothesis : } a_k = 3 \cdot 7^{k-1} \text{ "} \\ &= 3 \cdot 7^k \end{aligned}$$

and therefore $P(k+1)$ is true.

From (I) basis step and (II) inductive step, $P(n)$ is true for any integer $n \geq 1$ by mathematical induction proof. ■