H-D model

$Y = \frac{K}{V}$ where $V = \frac{K}{Y}$: ratio of capital to output.

ex: $Y_1 = \$100{,}000$, $K_1 = \$10m$. $\Rightarrow \frac{K_1}{Y_1} = \frac{10{,}000{,}000}{100{,}000} = 100:1$

$Y_2 = \$200{,}000$, $K_2 = \$20m$ $\Rightarrow \frac{K_2}{Y_2} = \frac{20{,}000{,}000}{200{,}000} = 100:1$

$\Rightarrow v = \frac{K}{Y} = 100:1.$

$\frac{\Delta K}{\Delta Y}$: incremental capital-output ratio

$\frac{\Delta K}{\Delta Y} = v \Rightarrow \Delta Y = \frac{1}{v}\cdot \Delta K$ or $\Delta K = v \cdot \Delta Y$

Recall

$$\Delta K = \overbrace{sY}^{= I = S = sY} - dK$$

$$g = \frac{\Delta Y}{Y} \quad \text{and} \quad \Delta Y = \frac{1}{v} \cdot \Delta K$$

$$g = \frac{1}{v} \cdot \Delta K \cdot \frac{1}{Y} = \frac{sY - dK}{vY}$$

$$g = \frac{s}{v} - \frac{d}{\cancel{v}} \cdot \overbrace{\frac{K}{Y}}^{= \cancel{v}}$$

$$\therefore \quad \boxed{g = \frac{s}{v} - d}$$

where g = output growth rate
s = saving rate
v = capital-output ratio ($v = K/Y$)
d = depreciation rate

Given $\Delta K = I - dK = sY - dK$

$\hookrightarrow$ $\Delta k = ?$ (k = capital per worker $= \frac{K}{L}$)

$$k = \frac{K}{L}$$

$$\ln(k) = \ln\left(\frac{K}{L}\right) = \ln(K) - \ln(L)$$

Totally differentiate:

$$\frac{1}{k} \cdot dk = \frac{1}{K} \cdot dK - \frac{1}{L} \cdot dL$$

$$\Rightarrow \frac{\Delta k}{k} = \frac{\Delta K}{K} - \underbrace{\frac{\Delta L}{L}}_{=n} = \frac{\Delta K}{K} - n$$

$$\therefore \frac{\Delta k}{k} = \frac{\Delta K}{K} - n$$

$$\frac{\Delta k}{k} = \frac{\overset{\Delta K}{(sY - dK)}}{K} - n$$

$$\frac{\Delta k}{k} = s \cdot \frac{Y}{K} - d - n$$

$$k = \frac{K}{L}$$

$$\Delta k = s \cdot \frac{Y}{K} \cdot k - (n+d)k$$

$$s \cdot \frac{Y}{\cancel{K}} \cdot \frac{\cancel{K}}{L}$$

$$\Delta k = s \cdot \frac{Y}{L} - (n+d)k$$

$$\boxed{\Delta k = s \cdot y - (n+d)k.}$$

Change in capital per worker

Solow Model

Given $y = k^{\alpha}$, $0 < \alpha < 1$, $y = \frac{Y}{L}$

✓ . $\Delta k = sy - (n+d)k$. $k = \frac{K}{L}$

Questions : 1) What is k_{ss}?

2) $\frac{\Delta y}{y} = ?$ and $\frac{\Delta Y}{Y} = ?$

At s.s. , $\Delta k = sy - (n+d)k = 0$

$$sy = (n+d)k$$

$$sk^{\alpha} = (n+d)k$$

$$k^{\alpha} = \left(\frac{n+d}{s}\right)k$$

$$k^{\alpha-1} = \left(\frac{n+d}{s}\right)$$

$$k_{ss} = \left(\frac{n+d}{s}\right)^{\frac{1}{\alpha-1}}$$

$$\boxed{k_{ss} = \left(\frac{s}{n+d}\right)^{\frac{1}{1-\alpha}}}$$

$$y_{ss} = \left(\frac{s}{n+d}\right)^{\alpha/(1-\alpha)}$$

What is the growth rate at S.S.?

At S.S., $\left.\begin{array}{l}\Delta k = 0 \\ y = k^{\alpha}\end{array}\right\} \Rightarrow \Delta y = 0$

↳ At S.S., growth of output per worker is zero.

$g = \frac{\Delta Y}{Y} = ?$

$$y = \frac{Y}{L}$$

$$\ln(y) = \ln(Y) - \ln(L)$$

$$\frac{dy}{y} = \frac{dY}{Y} - \frac{dL}{L}$$

$$\frac{\Delta y}{y} = \frac{\Delta Y}{Y} - n \Rightarrow \frac{\Delta Y}{Y} = \frac{\Delta y}{y} + n$$

At S.S., $\frac{\Delta y}{y} = 0$,

$$\frac{\Delta Y}{Y} = n$$