Assignment 2 EE320 (Section Aj. Kittichai)

Due on Nov., 8th 2020 (Sunday)

Instruction

1) Attempt all
2) To submit your homework, write your filename as follow **hw2_Group_0x**. One point will be deducted if you don't follow the format of suggested filename.

Question 1:

Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?
1.2 Is the production function increasing with respect to K and L?
1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.
1.4 Use the Hessian matrix. Proof that the production function is concave.

Question 2: Define f(x,y) for all (x,y) by

$$f\,(x, y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

2.1 Derive the Hessian matrix of $f\,(x, y)$.
2.2 Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).
2.3 Find the *global* extrema of f(x,y). What type of extrema is it?

Question 3:

A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and

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a) return to scale = degree of homogeneous

$Q = 18(0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$

$= (-0.4)(-2.5)$

$= 1$ Hence, the function is constant return to scale

b) $\frac{\partial Q}{\partial K} = -45(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-0.08K^{-1.4})$

$= 45(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(0.08K^{-1.4})$

$\frac{\partial Q}{\partial L} = -45(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-0.32L^{-1.4})$

$45(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(0.32L^{-1.4})$

Both are positive so the function is increasing with respect to both K and L

c) $Q = 18(0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$

$MRTS = -\frac{dK}{dL} = \frac{MPL}{MPK} = \frac{45(0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(0.32L^{-1.4})}{45(0.2K^{-0.4}+0.8L^{-0.4})^{-3.5}(0.08K^{-1.4})} = \frac{0.32L^{-1.4}}{0.08K^{-1.4}} = 4\frac{L^{-1.4}}{K^{-1.4}}$; MPL and MPK from $\frac{\partial Q}{\partial L}$ and $\frac{\partial Q}{\partial K}$ from question B

d) $|H_1| = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix} =$ where $(0.2K^{-0.4} + 0.8L^{-0.4}) = x$

$$\begin{bmatrix} 157.5(x)^{-4.5}(0.0064K^{-2.8}) - 45(x)^{-3.5}(0.112K^{-2.4}) & 157.5(x)^{-4.5}(-0.32L^{-1.4})(-0.08K^{-1.4}) \\ 157.5(x)^{-4.5}(-0.08K^{-1.4})(-0.32L^{-1.4}) & 157.5(x)^{-4.5}(0.1024L^{-2.8}) - 45(x)^{-3.5}(0.448L^{-2.4}) \end{bmatrix}$$

$|H_1| = \underbrace{157.5(x)^{-4.5}(0.0064K^{-2.8})}_{1} - \underbrace{45(x)^{-3.5}(0.112K^{-2.4})}_{2} < 0$ Since 2 is much larger than 1

$|H_2| = [157.5(x)^{-4.5}(0.0064K^{-2.8}) - 45(x)^{-3.5}(0.112K^{-2.4})][157.5(x)^{-4.5}(0.1024L^{-2.8}) - 45(x)^{-3.5}(0.448L^{-2.4})] - [157.5(x)^{-4.5}(-0.32L^{-1.4})(-0.08K^{-1.4})]^2 > 0$

So the function is negative definite so it is concave

Question 2: Define f(x,y) for all (x,y) by

$$f\ (x,y)\ =\ e^{\,x+y} + e^{\,x-y} - \tfrac{3}{2}\,x\ -\ \tfrac{1}{2}y$$

$f = e^{2x} - \frac{3}{2}x - \frac{1}{2}y$

2.1 Derive the Hessian matrix of $f\ (x,y)$.

2.2 Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).

2.3 Find the *global* extrema of f(x,y). What type of extrema is it?

a) $|H| : \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} e^{x+y}+e^{x-y} & e^{x+y}-e^{x-y} \\ e^{x+y}-e^{x-y} & e^{x+y}+e^{x-y} \end{bmatrix}$

b) $(e^{x+y}+e^{x-y})(e^{x+y}+e^{x-y}) = e^{2x+2y} + 2e^{2x} + e^{2x-2y}$ ①

$(e^{x+y}-e^{x-y})(e^{x+y}-e^{x-y}) = e^{2x+2y} - 2e^{2x} + e^{2x-2y}$ ②

① - ② : $4e^{2x}$

$|H|' = e^{x+y}+e^{x-y}$; both positive at the entire domain of (x,y) Hence the f² is monotonically convex

$|H|^2 = 4e^{2x}$

The function is positive definite, Hence the extrema is global minimizer

$f = e^{2x} - \frac{3}{2}x - \frac{1}{2}y$

$\frac{\partial f}{\partial x} = 2e^{2x} - \frac{3}{2}$

$\frac{\partial f}{\partial y} = -\frac{1}{2}$

a)

Alternative $|H| = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 4e^{2x} & 0 \\ 0 & 0 \end{bmatrix}$

b) $|H_1| = 4e^{2x} > 0$

$|H_2| = 4e^{2x} > 0$: both positive for entire domain of x, y. Hence f² is monotonically convex

c) $2e^{2x} - \frac{3}{2} = 0$

$2e^{2x} = \frac{3}{2}$

$e^{2x} = \frac{3}{4}$

$2x = \ln\frac{3}{4}$

$x = \frac{1}{2}\left(\ln\frac{3}{4}\right)$

$|H_1|$ = positive

$|H_2|$ = positive

Hence the function have globally minimize extrema the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

3.5) Confirm your answer with the second-order condition.

3.1) What type of the return to scale technology does the production function exhibit?

$$Q = K^{\frac{1}{3}} L^{\frac{2}{3}}$$

$$Q' = (tK)^{\frac{1}{3}} (tL)^{\frac{2}{3}}$$

$$= t^{\frac{1}{3}} K^{\frac{1}{3}} \cdot t^{\frac{2}{3}} L^{\frac{2}{3}}$$

$$= t^{\frac{3}{3}} \cdot K^{\frac{1}{3}} \cdot L^{\frac{2}{3}}$$

$\therefore$ degree of Homogeneity $= 1$

$Q(K,L)$ has a constant return to scale.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$\pi = P \cdot Q - \text{cost}$$

$$= Q^{-\frac{1}{4}} \cdot K^{\frac{1}{3}} L^{\frac{2}{3}} - (wL + rK)$$

$$= (K^{\frac{1}{3}} L^{\frac{2}{3}})^{-\frac{1}{4}} \cdot K^{\frac{1}{3}} L^{\frac{2}{3}} - wL - rK$$

$$= K^{\frac{3}{12}} L^{\frac{6}{12}} - wL + rK$$

$$\pi = K^{\frac{1}{4}} L^{\frac{1}{2}} - wL + rK \;\; *$$

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

$$\frac{d\pi}{du} = \frac{1}{4} \cdot L^{\frac{1}{2}} K^{-\frac{3}{4}} - r = 0$$

$$u^{-\frac{3}{4}} = \frac{4r}{\sqrt{L}}$$

$$K = \left(\frac{4r}{\sqrt{L}}\right)^{-\frac{3}{4}} \#$$

$$\frac{d\pi}{dL} = \frac{1}{2} \cdot K^{\frac{1}{4}} L^{-\frac{1}{2}} - w = 0$$

$$L^{-\frac{1}{2}} = \frac{2w}{\sqrt[4]{K}}$$

$$L = \left(\frac{2w}{\sqrt[4]{K}}\right)^{-\frac{1}{2}} \#$$

$$L \to K : K^* = \left(\frac{\left(\left(\frac{\sqrt[4]{K}}{2w}\right)^{2}\right)^{\frac{1}{2}}}{4r}\right)^{\frac{1}{3}} = \left(\frac{1}{8rw}\right)^{\frac{8}{9}} \#$$

$$u^* \to L^* = \left(\frac{\left(\left(\frac{1}{8rw}\right)^{\frac{8}{9}}\right)^{\frac{1}{4}}}{2w}\right)^{2} = \frac{1}{4w}\left(\frac{1}{8rw}\right)^{\frac{4}{9}} \#$$

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

$$\frac{dL^*}{dw} : \frac{1}{\cancel{4}w} \cdot (8r)^{-\frac{4}{9}} \cdot w^{-\frac{13}{9}} \cdot \frac{-\cancel{4}}{9} + \left(\frac{1}{8rw}\right)^{\frac{4}{9}} \cdot \frac{1}{4} \cdot -1 \cdot w^{-2}$$

$$\frac{-1}{9\left((8r)^{\frac{4}{9}} \cdot w^{\frac{22}{9}}\right)} + \frac{-1}{4\left((8r)^{\frac{4}{9}} \cdot w^{\frac{22}{9}}\right)}$$

$$\frac{-13}{36\left((8r)^{\frac{1}{9}} \cdot w^{\frac{22}{9}}\right)} \quad *$$

$$\frac{dL^*}{dr} : \frac{1}{4w}(8rw)^{-\frac{4}{9}}$$

$$\frac{1}{4w}(8w)^{-\frac{4}{9}} \cdot (r)^{-\frac{4}{9}}$$

$$\frac{1}{\cancel{4}w}(8w)^{-\frac{4}{9}} \cdot -\frac{\cancel{4}}{9} \cdot r^{-\frac{13}{9}}$$

$$\frac{-1}{9\left((8w)^{\frac{4}{9}} \cdot r^{\frac{13}{9}}\right)} \quad *$$

3.5) Confirm your answer with the second-order condition.

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \cdot L^{\frac{1}{2}} \cdot \frac{-3}{4} \cdot K^{-\frac{7}{4}} & \frac{1}{4} \cdot K^{-\frac{3}{4}} \cdot \frac{1}{2} \cdot L^{-\frac{1}{2}} \\ \frac{1}{2} \cdot \frac{1}{4} \cdot L^{-\frac{1}{2}} \cdot K^{-\frac{3}{4}} & \frac{1}{2} \cdot -\frac{1}{2} \cdot K^{\frac{1}{4}} \cdot L^{-\frac{3}{2}} \end{bmatrix}$$

$$|H_1| = \frac{1}{4} \cdot L^{\frac{1}{2}} \cdot \frac{-3}{4} \cdot K^{-\frac{7}{4}} = \frac{-3\sqrt{L}}{16K\sqrt[4]{K^3}}$$

$$|H_2| = \left(\frac{1}{4} \cdot L^{\frac{1}{2}} \cdot \frac{-3}{4} \cdot K^{\frac{-7}{4}} \cdot \frac{1}{2} \cdot -\frac{1}{2} \cdot u^{\frac{1}{4}} \cdot L^{\frac{-3}{2}}\right) - \left(\frac{1}{2} \cdot \frac{1}{4} \cdot L^{-\frac{1}{2}} \cdot u^{\frac{-3}{4}} \cdot \frac{1}{4} \cdot u^{\frac{-3}{4}} \cdot \frac{1}{2} \cdot L^{\frac{-1}{2}}\right)$$

$$= \frac{-7}{64 L K \sqrt{u}}$$

plug in $K = \left(\frac{4r}{\sqrt{L}}\right)^{\frac{-4}{3}}$, $L = \left(\frac{2w}{\sqrt[4]{K}}\right)^{-2}$ in $|H_1|$, $|H_2|$

$$|H_1| = \frac{-3\left(\left(\frac{2w}{\sqrt[4]{K}}\right)^{\cancel{-2}\,-1}\right)^{\cancel{\frac{1}{2}}\,1}}{16 \cdot \left(\frac{4r}{\sqrt{L}}\right)^{\frac{-4}{3}} \cdot \left(\left(\frac{4r}{\sqrt{L}}\right)^{-\frac{\cancel{4}}{\cancel{3}}}\right)^{\frac{\cancel{3}\,1}{\cancel{4}\,1}}}$$

$$|H_1| = \frac{-3\left(\frac{\sqrt[4]{K}}{2w}\right)}{16 \cdot \left(\frac{\sqrt{L}}{4r}\right)^{\frac{7}{3}}} < 0$$

$$|H_2| = \frac{-7}{64 \cdot \left(\frac{4r}{\sqrt{L}}\right)^{\frac{-4}{3}} \cdot \left(\frac{2w}{\sqrt[4]{K}}\right)^{-2} \cdot \left(\left(\frac{2w}{\sqrt[4]{K}}\right)^{\cancel{-2}}\right)^{\frac{1}{\cancel{2}\,1}}}$$

$$= \frac{-7}{64 \left(\frac{\sqrt{L}}{4r}\right)^{\frac{4}{3}} \cdot \left(\frac{\sqrt[4]{u}}{2w}\right)^{3}} < 0$$

H is negative definite at K, L ; $d^2\pi < 0$; at u, L is local maximizer

$\therefore$ π is globally concave.