Chapter 3 Solving Systems

- **Systems of Linear Equations**

In this chapter we will learn how to solve the systems of linear equations by the method of Gaussian Elimination. In this class we only consider the systems which the number of unknown variables is the same as the number of equations.

Linear equations system is a system that contains several equations. We write the linear equations system with *n* equations and *n* variables as

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{aligned}$$

where $a_{11}, a_{12}, ..., a_{nn}, b_1, b_2, ..., b_n$ are real numbers and $x_1, x_2, ..., x_n$ are variables. The variables $x_1, x_2, ..., x_n$ which satisfy the *n* linear equations system are called the ***solution*** of the system.

Gaussian Elimination

There are several methods to solve the linear systems but in this chapter we will only consider the method of Gaussian Elimination.

There are three types of the operation.

1. Interchanging row *i* and row *j*, denoted by $R_i \leftrightarrow R_j$.
2. Multiplying row *i* by a constant *k* where $k \neq 0$, denoted by $kR_i \to R_i$.
3. The summation of row *i* and *k* times row *j* where $k \neq 0$, denoted by $R_i + kR_j \to R_i$.

We use the operations above to solve for the solution of linear equations system by eliminating the variables until the linear equations system has the following form:

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \ldots + c_{2n}x_n &= d_2 \\ &\vdots \\ c_{nn}x_n &= d_n \end{aligned}$$

Example 1: Find the solution of the linear equations system

$$
\begin{aligned}
4x_1 - 2x_2 + 3x_3 &= 1 \\
-x_1 + 3x_2 + x_3 &= 10 \\
2x_1 + x_2 - x_3 &= -3
\end{aligned}
$$

Example 2: Find the solution of the linear equations system

$$
\begin{aligned}
2x_1 - x_2 + 3x_3 + 3x_4 &= 3 \\
4x_1 \qquad + 2x_3 - 9x_4 &= -1 \\
3x_2 + 5x_3 \qquad &= -3 \\
4x_2 \qquad + 6x_4 &= -2
\end{aligned}
$$

Example 3: Consider the linear equations system

$$
\begin{aligned}
2x_1 \quad + 3x_3 &= -1 \\
x_1 + 2x_2 - x_3 &= 1 \\
\beta x_2 - 5x_3 &= -2
\end{aligned}
$$

Find the constant β for which the linear equations system has no solution.

Example 4: A chemical manufacturer wishes to fill an order for 800 gallons of 25% acid solution. Solutions of 20% and 35% are in stock. How many gallons of each solution must be mixed to fill the order?

Nonlinear Systems

A system of equations in which at least one equation is not linear is called a nonlinear system.

Strategy: If a nonlinear system contains a linear equation, we usually solve the linear equation for one variable and substitute for that variable in the other equation.

Example 5: Solve the system

$$\begin{aligned} x^2 - 2x + y - 7 &= 0 \\ 3x - y + 1 &= 0 \end{aligned}$$

Example 6: Solve the system

$$\begin{aligned} y &= \sqrt{x+2} \\ x + y &= 4 \end{aligned}$$