

Solution: Assignment 4 (Part II)

1. Let x and y be real numbers with $|x| < \frac{1}{2}$. Show that $|xy - x| < \frac{|y|+1}{2}$.

Answer: From $|x| < \frac{1}{2}$,

$$|xy - x| = |x(y - 1)| = |x||y - 1| < \frac{1}{2}|y - 1| \leq \frac{1}{2}(|y| + 1).$$

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2. Let x be a real number with $|x| \leq 3$. Determine if the following inequality is true or not.

$$|x^2 - 4| \geq 5|x + 2|$$

Explain your answer.

Answer: False. First note that, for $|x| \leq 3$, $|x - 2| \leq |x| + 2 \leq 3 + 2 = 5$. That is,

$$|x - 2| \leq 5.$$

From $x^2 - 4 = (x + 2)(x - 2)$ and from $|x - 2| \leq 5$,

$$|x^2 - 4| = |(x - 2)(x + 2)| = |(x - 2)||x + 2| \leq 5|x + 2|.$$

That is, $|x^2 - 4| \leq 5|x + 2|$. Therefore, the given inequality is false.

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3. Let x and y be real numbers with $x > 1$ and $y < -2$. Determine if the following inequality is true or not.

$$1 + \frac{|y|}{x} > \frac{4 - y}{2|x|}$$

Explain your answer.

Answer: True. Since $x > 1$ and $y < -2$, we have $x > 0$ and $y < 0$. This implies $x = |x|$ and $y = -|y|$. I.e., the above inequality is equivalent to

$$1 - \frac{y}{x} > \frac{4 - y}{2x}.$$

Since $x > 0$, we have

$$2x \left(1 - \frac{y}{x}\right) > 4 - y \quad \Leftrightarrow \quad 2x - 2y > 4 - y \quad \Leftrightarrow \quad 2x - 2 > 2 + y.$$

That is, the given inequality is true if and only if $2x - 2 > 2 + y$ is true. Notice that, since $x > 1$,

$$2x - 2 = 2(x - 1) > 0$$

and since $y < -2$,

$$2 + y < 0.$$

That is, $2x - 2$ is positive and $2 + y$ is negative. Hence, it is true that

$$2x - 2 > 2 + y.$$

and therefore the given inequality is true.

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4. Find the solution set for each of following inequalities.

(a) $1 < |x - 1| \leq 6$

Answer: There are two possible ways to solve this problem.

Method I By definition,

$$|x - 1| = \begin{cases} x - 1, & x \geq 1 \\ -(x - 1), & x < 1 \end{cases}$$

We consider 2 cases for $|x - 1|$: $x \geq 1$ and $x < 1$.

Case I: $x \in [1, \infty)$: For $x \geq 1$, $1 < |x - 1| \leq 6$ becomes

$$\begin{array}{rcl} 1 & < x - 1 \leq & 6 \\ 1 + 1 & < x \leq & 6 + 1 \\ 2 & < x \leq & 7 \end{array}$$

or $x \in (2, 7]$. That is, the solution set for this case is $[1, \infty) \cap (2, 7] = (2, 7]$.

Case II: $x \in (-\infty, 1)$: For $x < 1$, $1 < |x - 1| \leq 6$ becomes

$$\begin{array}{rcl} 1 & < -x + 1 \leq & 6 \\ 1 - 1 & < -x \leq & 6 - 1 \\ 0 & > x \geq & -5 \end{array}$$

or $x \in [-5, 0)$. That is, the solution set for this case is $(-\infty, 1) \cap [-5, 0) = [-5, 0)$.

From cases I and II, the solution set is $[-5, 0) \cup (2, 7]$. ■

Method II We can consider the given statement $1 < |x - 1| \leq 6$ as two inequalities:

$1 < |x - 1|$ and $|x - 1| \leq 6$

(i) For $1 < |x - 1|$, we have either $x - 1 > 1$ or $x - 1 < -1$.

That is, $x - 1 > 1 \Rightarrow x > 2$ or $x - 1 < -1 \Rightarrow x < 0$.

I.e. $x \in (-\infty, 0) \cup (2, \infty)$.

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(ii) For $|x - 1| \leq 6$, we have $-6 \leq x - 1 \leq 6$.

That is,

$$\begin{array}{rcl} -6 & \leq x - 1 \leq & 6 \\ -6 + 1 & \leq x \leq & 6 + 1 \\ -5 & \leq x \leq & 7 \end{array}$$

or $x \in (-5, 7)$.

Since both (i) and (ii) have to be true, the solution set is

$$\{(-\infty, 0) \cup (2, \infty)\} \cap (-5, 7) = [-5, 0) \cup (2, 7].$$

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(b)

$$\frac{(|x| + 5)(x - 1)}{|x - 2| + x^2 + x + 1} \geq 0$$

Answer: First notice that, $x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$ (because for $a = 1, b = 1, c = 1$, $b^2 - 4ac = 1 - 4 = -3 < 0$ and $a = 1 > 0$). Also notice that $|x - 2| \geq 0$ for all $x \in \mathbb{R}$. That is, the denominator

$$|x - 2| + x^2 + x + 1 > 0$$

for all $x \in \mathbb{R}$. Therefore, in order to have $\frac{(|x|+5)(x-1)}{|x-2|+x^2+x+1} \geq 0$, we only need to consider when

$$(|x| + 5)(x - 1) \geq 0, \text{ which can be separated into 2 subcases by using } |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}.$$

Case I When $x < 0$,

$$(|x| + 5)(x - 1) = (-x + 5)(x - 1) = -(x - 5)(x - 1).$$

So we find x such that $-(x - 5)(x - 1) \geq 0$ or $(x - 5)(x - 1) \leq 0$. By setting $(x - 5)(x - 1) = 0$, we consider $x = 5$ and $x = 1$ to construct 3 subintervals.

	$x \in (-\infty, 1)$	$x \in (1, 5)$	$x \in (5, \infty)$
$x - 1$	-	+	+
$x - 5$	-	-	+
$(x - 5)(x - 1)$	+	-	+

So, $x \in [1, 5]$ I.e. the solution set of $(|x| + 5)(x - 1) \geq 0$ for this case is $(-\infty, 0) \cap [1, 5] = \emptyset$.

Remark: It is also possible to argue that, since

$$x < 0 \Rightarrow x - 1 < 0$$

and

$$|x| \geq 0 \Rightarrow |x| + 5 > 0 \Rightarrow (|x| + 5) > 0$$

for any $x \in \mathbb{R}$, then the product $(|x| + 5)(x - 1)$ must always be negative and it is impossible to have $(|x| + 5)(x - 1) \geq 0$ and the solution set for this case is \emptyset .

Case II When $x \geq 0$,

$$(|x| + 5)(x - 1) = (x + 5)(x - 1) = (x + 5)(x - 1).$$

So we find x such that $(x + 5)(x - 1) \geq 0$. By setting $(x + 5)(x - 1) = 0$, we consider $x = -5$ and $x = 1$ to construct 3 subintervals.

	$x \in (-\infty, -5)$	$x \in (-5, 1)$	$x \in (1, \infty)$
$x + 5$	-	+	+
$x - 1$	-	-	+
$(x + 5)(x - 1)$	-	-	+

That is, $x \in (-\infty, -5] \cup [1, \infty)$. I.e. the solution set of $(|x| + 5)(x - 1) \geq 0$ for this case is $[0, \infty) \cap \{(-\infty, -5] \cup [1, \infty)\} = [1, \infty)$.

From cases I and II, the solution set is $\emptyset \cup [1, \infty) = [1, \infty)$. ■

(c)

$$2x + \frac{1}{|x|} \geq 1$$

Answer: $[-\frac{1}{2}, 0) \cup (0, \infty)$. From $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$, we will consider 2 cases.

Case I: $x \in (-\infty, 0)$ When $x < 0$, $|x| = -x$ and the given inequality becomes

$$\begin{aligned} 2x - \frac{1}{x} &\geq 1 \\ 2x^2 - x - 1 &\geq 0 \\ (2x + 1)(x - 1) &\geq 0. \end{aligned}$$

By setting $(2x + 1)(x - 1) = 0$, we consider $x = -1/2$ and $x = 1$ to construct 3 subintervals.

	$x \in (-\infty, -1/2)$	$x \in (-1/2, 1)$	$x \in (1, \infty)$
$2x + 1$	-	+	+
$x - 1$	-	-	+
$(2x + 1)(x - 1)$	+	-	+

That is, $x \in (-\infty, -1/2] \cup [1, \infty)$. I.e. the solution set for this case is $(-\infty, 0) \cap \{(-\infty, -1/2] \cup [1, \infty)\} = [-1/2, 0)$.

Case II: $x \in [0, \infty)$ When $x \geq 0$, we first note that we must have $x \neq 0$ because x is the denominator. For $x > 0$, $|x| = x$ and the given inequality becomes

$$\begin{aligned} 2x + \frac{1}{x} &\geq 1 \\ 2x^2 - x + 1 &\geq 0. \end{aligned}$$

For $a = 2, b = 1, c = 1$, $b^2 - 4ac = 1 - 8 < 0$, so we cannot factor $2x^2 - x + 1$. Since $a > 0$, then $2x^2 - x + 1 > 0$ for all $x \in \mathbb{R}$. Recall $x \neq 0$. That is, the solution set for this case is $[0, \infty) \cap \mathbb{R} - \{0\} = (0, \infty)$.

From cases I and II, the solution set is $[-1/2, 0) \cup (0, \infty)$. ■

(d)

$$\frac{|2 - x|}{x^3 + 3x^2 + 11x + 18} \leq 0$$

Answer: First notice that

$$x^3 + 3x^2 + 11x + 18 = \underbrace{x^3 + 2x^2}_{x^2(x+2)} + \underbrace{x^2 + 11x + 18}_{(x+9)(x+2)} = x^2(x+2) + (x+9)(x+2) = (x^2 + x + 9)(x+2).$$

For $a = 1, b = 1, c = 9, b^2 - 4ac = 1 - 36 < 0$, so we cannot factor $x^2 + x + 9$. Since $a > 0$, then $x^2 + x + 9 > 0$ for all $x \in \mathbb{R}$. That is,

$$\frac{|2 - x|}{x^3 + 3x^2 + 11x + 18} = \frac{|2 - x|}{(x^2 + x + 9)(x + 2)}$$

and since $x^2 + x + 9 > 0$ for all $x \in \mathbb{R}$, the solution set of $\frac{|2-x|}{x^3+3x^2+11x+18} \leq 0$ is the same as the one for

$$\frac{|2 - x|}{x + 2} \leq 0.$$

Two approaches for solving this will be presented here (short and long approaches).

(i) **Short Approach** Since $|2 - x| \geq 0$ (by property of absolute value) for all $x \in \mathbb{R}$, then we can only have $\frac{|2-x|}{x+2} \leq 0$ when $x + 2 < 0$. That is, $x < -2$ or the solution set for x is $(-\infty, -2)$. ■

(i) **Long Approach** From $|2 - x| = \begin{cases} -(2 - x), & 2 - x < 0 \Leftrightarrow x > 2 \\ 2 - x, & 2 - x \geq 0 \Leftrightarrow x \leq 2 \end{cases}$, we will consider 2 cases.

Case I: $x \in (2, \infty)$ We have $|2 - x| = x - 2$ and $\frac{|2-x|}{x+2} = \frac{x-2}{x+2}$ So we want to solve

$$\frac{x - 2}{x + 2} \leq 0$$

Consider the intervals divided by $x = -2, x = 2$.

	$x \in (-\infty, -2)$	$x \in (-2, 2)$	$x \in (2, \infty)$
$x + 2$	-	+	+
$x - 2$	-	-	+
$\frac{x+2}{x-2}$	+	-	+

To have non-zero denominator, we must have $x \neq -2$. Since we have “less than or equal” sign, then $x \in (-2, 2]$. Hence, the solution set is $(2, \infty) \cap (-2, 2] = \emptyset$.

Case II: $x \in (-\infty, 2]$ We have $|2 - x| = 2 - x$ So we want to solve

$$\frac{2 - x}{x + 2} \leq 0 \quad \text{or} \quad \frac{x - 2}{x + 2} \geq 0$$

Consider the intervals divided by $x = -2, x = 2$.

	$x \in (-\infty, -2)$	$x \in (-2, 2)$	$x \in (2, \infty)$
$x + 2$	-	+	+
$x - 2$	-	-	+
$\frac{x+2}{x-2}$	+	-	+

To have non-zero denominator, we must have $x \neq -2$. Since we have “less than or equal” sign, then $x \in (-\infty, -2) \cup (2, \infty)$. Hence, the solution set is $(-\infty, 2] \cap \{(-\infty, -2) \cup (2, \infty)\} = (-\infty, -2)$

From cases I and II, $\emptyset \cup (-\infty, -2) = (-\infty, -2)$ ■

(e)

$$\frac{x^2 + 1 - |x - 1|}{5 - |x + 3|} \leq 0$$

Answer:

From $|x-1| = \begin{cases} -(x-1), & x-1 < 0 \Leftrightarrow x < 1 \\ x-1, & x-1 \geq 0 \Leftrightarrow x \geq 1 \end{cases}$, and $|x+3| = \begin{cases} -(x+3), & x+3 < 0 \Leftrightarrow x < -3 \\ x+3, & x+3 \geq 0 \Leftrightarrow x \geq -3 \end{cases}$, we will consider 3 cases:

Case I: $x \in (-\infty, -3)$, Case II: $x \in [-3, 1)$, Case III: $x \in [1, \infty)$.

Case I: $x \in (-\infty, -3)$, $|x - 1| = -(x - 1)$, $|x + 3| = -(x + 3)$ and

$$\frac{x^2 + 1 - |x - 1|}{5 - |x + 3|} = \frac{x^2 + 1 + x - 1}{5 + x + 3} = \frac{x^2 + x}{x + 8} = \frac{x(x + 1)}{x + 8}.$$

So we will solve $\frac{x(x+1)}{x+8} \leq 0$. To obtain the subintervals, consider $x = -8, -1, 0$

	$x \in (-\infty, -8)$	$x \in (-8, -1)$	$x \in (-1, 0)$	$x \in (0, \infty)$
$x + 8$	-	+	+	+
$x + 1$	-	-	+	+
x	-	-	-	+
$\frac{x(x+1)}{x+8}$	-	+	-	+

To have non-zero denominator, we must have $x \neq -8$. Since we have “less than or equal” sign, then the solution set is $(-\infty, -8) \cup [-1, 0]$. I.e., the solution set is $(-\infty, -3) \cap \{(-\infty, -8) \cup [-1, 0]\} = \boxed{(-\infty, -8)}$.

Case II: $x \in [-3, 1)$, $|x - 1| = -(x - 1)$, $|x + 3| = x + 3$ and

$$\frac{x^2 + 1 - |x - 1|}{5 - |x + 3|} = \frac{x^2 + 1 + x - 1}{5 - x - 3} = \frac{x^2 + x}{-x + 2} = -\frac{x(x + 1)}{x - 2}.$$

So we will solve $-\frac{x(x+1)}{x-2} \leq 0$ or $\frac{x(x+1)}{x-2} \geq 0$. To obtain the subintervals, consider $x = -1, 0, 2$

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (-1, 2)$	$x \in (2, \infty)$
$x + 1$	-	+	+	+
x	-	-	+	+
$x - 2$	-	-	-	+
$\frac{x(x+1)}{x-2}$	-	+	-	+

To have non-zero denominator, we must have $x \neq 2$. Since we have “less than or equal” sign, then $x \in [-1, 0] \cup (2, \infty)$. I.e., the solution set is $[-3, 1) \cap \{x \in [-1, 0] \cup (2, \infty)\} = \boxed{[-1, 0]}$.

Case III: $x \in [1, \infty)$, $|x - 1| = x - 1$, $|x + 3| = x + 3$ and

$$\frac{x^2 + 1 - |x - 1|}{5 - |x + 3|} = \frac{x^2 + 1 - x + 1}{5 - x - 3} = \frac{x^2 - x + 1}{-x + 2} = -\frac{x^2 - x + 1}{x - 2}.$$

So we want to solve

$$\frac{x^2 - x + 1}{x - 2} \geq 0.$$

For $a = 1, b = -1, c = 1, b^2 - 4ac = 1 - 4 < 0$, so we cannot factor $x^2 - x + 1$. Since $a > 0$, then $x^2 - x + 1 > 0$ for all $x \in \mathbb{R}$. So, in order to have $\frac{x^2 - x + 1}{x - 2} \geq 0$, we can consider instead $x - 2 > 0$ or $x \in (2, \infty)$. That is, the solution set for this case is $[1, \infty) \cap (2, \infty) = \boxed{(2, \infty)}$.

From cases I, II, III, the solution set is $(-\infty, -8) \cup [-1, 0] \cup (2, \infty)$. ■

5. (Optional) Find the solution set for each of the following inequities.

(a) $|x - 1| < 7$

Ans: $(-6, 8)$

(b) $3 < |x + 2| < 4$

Ans: $(-6, -5) \cup (1, 2)$

(c) $|2x + 3| \leq 2$

Ans: $[-5/2, -1/2]$

(d) $|3x - 1| \geq 1$

Ans: $(-\infty, 0] \cup [2/3, \infty)$

(e) $\frac{|x+2|}{2} > |x|$

Ans: $(-2/3, 2)$

(f) $|x + 1| - |2x - 1| < 0$

Ans: $(-\infty, 0) \cup (2, \infty)$

(g) $|2 - x| - x \leq 0$

Ans: $[1, \infty)$

(h) $|x + 1| > 3 - |x|$

Ans: $(-\infty, -2) \cup (1, \infty)$

(i) $x^2 - 2 \geq \frac{1}{2}|x - 1|$

Ans: $(-\infty, \frac{-1-\sqrt{41}}{4}) \cup (3/2, \infty)$

(j) $|x + 3| \geq 2 - x$

Ans: $[-1/2, \infty)$

(k) $|2x^2 + x - 1| \geq 2$

Ans: $(-\infty, -3/2] \cup [1, \infty)$

(l) $\frac{|x|}{2} + 3 > x^2$

Ans: $(-2, 2)$

(m) $|x| > \frac{2}{|x+1|}$

Ans: $(-\infty, -2) \cup (1, \infty)$

(n) $\frac{x^2}{x+1} < |x|$

Ans: $(-\infty, -1) \cup (-1/2, 0) \cup (0, \infty)$

(o) $\frac{4}{|x|} \leq \frac{1}{|x|}$

Ans: \emptyset

(p) $|x^2 + 1| < |x + 1|$

Ans: $(0, 1)$

(q) $(x^2 + 1)(|x + 2| - |x|) \geq 0$

Ans: $[-1, \infty)$

(r) $\frac{|1-x|}{x^3+2x^2+5x+4} \leq 0$

Ans: $(-\infty, -1) \cup \{1\}$