Question 1:

Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?

Members

- Purin Chantaravari 6204640954
- Wanrada kliebphipat 6204641200
- Gulanut Kiatsuranon 6204641416
- Hyeseong Yoo 6204641606

1.1 $Q = f(K, L) = 18 \cdot (0.2k^{-0.4} + 0.8L^{-0.4})^{-2.5}$

k_0, L_0 ; $Q_0 = 18\,(0.2\,k_0^{-0.4} + 0.8\,L_0^{-0.4})^{-2.5}$

tk_0, tL_0 ; $Q = 18\,(0.2\,(tk_0)^{-0.4} + 0.8\,(tL_0)^{-0.4})^{-2.5}$

$Q = (t^{-0.4})^{-2.5}\, 18\,(0.2\,k_0^{-0.4} + 0.8L_0^{-0.4})^{-2.5}$

$Q = t^1 Q_0$

constant return of scale $= 1$ #

1.2 Is the production function increasing with respect to K and L?

1.2 $\dfrac{dQ}{dk} = 18\,(-2.5)\,[\,0.2k^{-0.4} + 0.8L^{-0.4}\,]^{-3.5}\,(-0.08k^{-1.4}) > 0$

$\dfrac{dQ}{dL} = 18\,(-2.5)\,[\,0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}\,(-0.32L^{-1.4}) > 0$

$\therefore$ The production function increasing with respect to k and L #

1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.

1.3 $f(K, L) = 0$

$f(k, L) = 18\,(0.2k^{-0.4} + 0.8L^{-0.4})^{-2.5} = 0$

$$MRTS = -\frac{F_L}{F_k} \quad \frac{\cancel{18\,(-2.5)\,[\,0.2k^{-0.4} + 0.8L^{-0.4}\,]^{-3.5}}\,(-0.08k^{-1.4})}{\cancel{18\,(-2.5)\,[\,0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}}\,(-0.32L^{-1.4})}$$

$$= \frac{4L^{-1.4}}{K^{-1.4}}$$

#

1.4 Use the Hessian matrix. Proof that the production function is concave.

$$H = \begin{bmatrix} f_{kk} & f_{kL} \\ f_{Lk} & f_{LL} \end{bmatrix}$$

$$f_{kk} = \frac{\partial f}{\partial k} = 18(-2.5)\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-0.08k^{-1.4})$$

$$= (3.6k^{-1.4})(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5}$$

$$\frac{\partial f_k}{\partial k} = (3.6k^{-1.4})(-3.5)\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-4.5}(-0.08k^{-1.4}) + (0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5}(-5.04k^{-2.4})$$

$$= 1.008k^{-2.8}\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-4.5} + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-5.04k^{-2.4})$$

$$f_{kL} = \frac{\partial f}{\partial k} = (3.6k^{-1.4})(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5}$$

$$\frac{\partial f_k}{\partial L} = (3.6k^{-1.4})(-3.5)(0.2k^{-0.4} + 0.8L^{-0.4})^{-4.5}(-0.32L^{-1.4}) + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(0)$$

$$= (4.032k^{-1.4}L^{-1.4})\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-4.5}$$

$$f_{Lk} = \frac{\partial f}{\partial L} = 18(-2.5)\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-0.32L^{-1.4})$$

$$= (14.4L^{-1.4})(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5}$$

$$\frac{\partial f_L}{\partial k} = (14.4L^{-1.4})(-3.5)(0.2k^{-0.4} + 0.8L^{-0.4})^{-4.5}(-0.08k^{-1.4}) + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(0)$$

$$= (4.032k^{-1.4}L^{-1.4})\left[0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-4.5}$$

$$f_{LL} = \frac{\partial f}{\partial L} = (14.4L^{-1.4})(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5}$$

$$\frac{\partial f_L}{\partial L} = (14.4L^{-1.4})(-3.5)(0.2k^{-0.4} + 0.8L^{-0.4})^{-4.5}(-0.32L^{-1.4}) + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-20.16L^{-2.4})$$

$$= (16.128L^{-2.8})(0.2k^{-0.4} + 0.8L^{-0.4})^{-4.5} + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-20.16L^{-2.4})$$

$$|H_1| = \left|(1.008k^{-2.8})\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-4.5} + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-5.04k^{-2.4})\right|$$

$$= (1.008k^{-2.8})\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-4.5} + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-5.04k^{-2.4})$$

$$|H_2| = \left\{1.008k^{-2.8}\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-4.5} + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-5.04k^{-2.4})\right\} \times \left\{(16.128L^{-2.8})(0.2k^{-0.4} + 0.8L^{-0.4})^{-4.5} + \left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}(-20.16L^{-2.4})\right\} -$$

$$\left\{(4.032k^{-1.4}L^{-1.4})\left[0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-4.5}\right\} \times \left\{(4.032k^{-1.4}L^{-1.4})\left[0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-4.5}\right\}$$

$\therefore$ $|H_1| < 0$; $\forall k, \forall L$

$|H_2| > 0$; $\forall k, \forall L$

H is negative definite due to $d^2Q < 0$. the production function is concave.

2.1) $f_x = \frac{\partial f}{\partial x} = e^{x+y} + e^{x-y} - \frac{3}{2}$

$f_y = \frac{\partial f}{\partial y} = e^{x+y} - e^{x-y} - \frac{1}{2}$

$f_{xx} = e^{x+y} + e^{x-y}$
$f_{xy} = e^{x+y} - e^{x-y}$
$f_{yx} = e^{x+y} - e^{x-y}$
$f_{yy} = e^{x+y} + e^{x-y}$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} e^{x+y}+e^{x-y} & e^{x+y}-e^{x-y} \\ e^{x+y}-e^{x-y} & e^{x+y}+e^{x-y} \end{bmatrix}$$

2.2) $$H = \begin{bmatrix} e^{x+y}+e^{x-y} & e^{x+y}-e^{x-y} \\ e^{x+y}-e^{x-y} & e^{x+y}+e^{x-y} \end{bmatrix}$$

$|H_1| = |e^{x+y} + e^{x-y}| = e^{x+y} + e^{x-y} > 0$

$$|H_2| = \begin{vmatrix} e^{x+y}+e^{x-y} & e^{x+y}-e^{x-y} \\ e^{x+y}-e^{x-y} & e^{x+y}+e^{x-y} \end{vmatrix}$$

$= (e^{x+y}+e^{x-y})(e^{x+y}+e^{x-y}) - (e^{x+y}-e^{x-y})(e^{x+y}-e^{x-y})$
$= (e^{x+y}+e^{x-y})^2 - (e^{x+y}-e^{x-y})^2 > 0 \quad \forall x, \forall y$

$|H_i| > 0 \quad \forall x, \forall y$

H is positive definite

$d^2f > 0$, for all (x,y) $\rightarrow$ f(x,y) is monotonically convex function

2.3) 2 step process

step 1 $\rightarrow f_x = f_y = 0$

$f_x = \frac{\partial f}{\partial x} = e^{x+y} + e^{x-y} - \frac{3}{2} = 0$ — ①

$f_y = \frac{\partial f}{\partial y} = e^{x+y} - e^{x-y} - \frac{1}{2} = 0$ — ②

$e^{x+y} + e^{x-y} = \frac{3}{2}$ ①

$e^{x+y} - e^{x-y} = \frac{1}{2}$ ②

①+② : $2e^{x+y} = 2$
$e^{x+y} = 1$
$e^{x+y} = e^0$
$x+y = 0$
$x = -y$

①: $e^{-y+y} + e^{-y-y} = \frac{3}{2}$
$e^0 + e^{-2y} = \frac{3}{2}$
$e^{-2y} = \frac{1}{2}$
$-2y = \ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$

$\left.\begin{array}{l} y = \frac{\ln 2}{2} \\ \rightarrow x = -\frac{\ln 2}{2} \end{array}\right\} \left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right) \rightarrow$ critical point

Step 2: from 2.2

all $|H_i| > 0 \quad \forall i$

H is always positive definite.

$\rightarrow d^2f > 0$ at the critical point

$\rightarrow f$ is convex at the critical point

$\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is the local minimizer.

since $f(x,y)$ is monotonically convex function.

$\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is the global minimizer.

$\therefore$ The global minimum is:

$$f\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right) = e^{-\frac{\ln 2}{2} + \frac{\ln 2}{2}} + e^{-\frac{\ln 2}{2} - \frac{\ln 2}{2}} - \frac{3}{2}\left(-\frac{\ln 2}{2}\right) - \frac{1}{2}\left(\frac{\ln 2}{2}\right)$$

$$= e^0 + e^{-\ln 2} + \frac{3\ln 2}{4} - \frac{\ln 2}{4}$$

$$= 1 + \frac{1}{e^{\ln 2}} + \frac{\ln 2}{2}$$

$$= 1 + \frac{1}{2} + \frac{\ln 2}{2} = \frac{3}{2} + \frac{\ln 2}{2}$$

Question 3:

A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

$k_0, L_0 \quad : \quad Q = k^{\frac{1}{3}} L^{\frac{2}{3}}$

take $t \to tk_0, tL_0 \quad : \quad Q = (tk)^{\frac{1}{3}} (tL)^{\frac{2}{3}}$

$= t^{\frac{1}{3}} k^{\frac{1}{3}} \times t^{\frac{2}{3}} L^{\frac{2}{3}}$

$= t^{1} \times (k^{\frac{1}{3}} L^{\frac{2}{3}})$

$\therefore$ HM # 1

$\therefore$ So, the production function exhibit as a constant return to scale.

From now on, assume that c = ¼ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

Profit = Revenue − Cost

$= (P \times Q) - (wL + rk)$

$= Q^{-c} \cdot Q - wL - rk$

assume $c = \frac{1}{4}$: $= Q^{-\frac{1}{4}} \cdot Q - wL - rk$

$= Q^{\frac{3}{4}} - wL - rk$

assume $Q = k^{\frac{1}{3}} L^{\frac{2}{3}}$: $= (k^{\frac{1}{3}} L^{\frac{2}{3}})^{\frac{3}{4}} - wL - rk$

$\pi(k, L) = (k^{\frac{1}{4}} L^{\frac{1}{2}}) - wL - rk$

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

- $$\frac{\partial \pi}{\partial k} = \frac{1}{4} L^{\frac{1}{2}} k^{-\frac{3}{4}} - r$$

$$0 = \frac{1}{4} L^{\frac{1}{2}} k^{-\frac{3}{4}} - r$$

$$k^{-\frac{3}{4}} = \frac{4r}{L^{\frac{1}{2}}}$$

$$k^* = \left(\frac{4r}{L^{\frac{1}{2}}}\right)^{-\frac{4}{3}}$$

- $$\frac{\partial \pi}{\partial L} = \frac{1}{2} k^{\frac{1}{4}} L^{-\frac{1}{2}} - w$$

$$0 = \frac{1}{2} k^{\frac{1}{4}} L^{-\frac{1}{2}} - w$$

$$L^{-\frac{1}{2}} = \frac{2w}{k^{\frac{1}{4}}}$$

$$L^* = \left(\frac{2w}{k^{\frac{1}{4}}}\right)^{-2}$$

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

$$L^* = \left(\frac{2w}{k^{\frac{1}{4}}}\right)^{-2}$$

$$= \frac{(2w)^{-2}}{(k^{\frac{1}{4}})^{-2}}$$

$$= \frac{-4w^{-2}}{k^{-\frac{1}{2}}}$$

$$= -4w^{-2} k^{-\frac{1}{2}}$$

$$k^* = \left(\frac{4r}{L^{\frac{1}{2}}}\right)^{-\frac{4}{3}}$$

$$k^* = \frac{4^{-\frac{4}{3}} r^{-\frac{4}{3}}}{L^{-\frac{2}{3}}}$$

$$k^* = \frac{1}{4^{\frac{4}{3}} r^{\frac{4}{3}} L^{-\frac{2}{3}}}$$

$$k^* = \frac{1}{4 r^{4/3}}$$

→ Sub k^* into L^*

$$L^* = -4w^{-2} (4r^{\frac{4}{3}})^{\frac{1}{2}}$$

$$= -4w^{-2} (2r^{\frac{2}{3}})$$

$$= -8w^{-2} r^{\frac{2}{3}}$$

→ $$\frac{\partial L^*}{\partial w} = -8(-2) w^{-3} r^{\frac{2}{3}}$$

$$= -16w^{-3} r^{\frac{2}{3}}$$

∴ if the wages increase by 1 unit, the Labour will decrease by $16w^{-3} r^{\frac{2}{3}}$ units

→ $$\frac{\partial L^*}{\partial r} = -8w^{-2}\left(\frac{2}{3}\right) r^{-\frac{1}{3}}$$

$$= -\frac{16}{3} w^{-2} r^{-\frac{1}{3}}$$

∴ if the capital increase by 1 unit, the Labour will decrease by $\frac{16}{3} w^{-2} r^{-\frac{1}{3}}$ units

3.5) Confirm your answer with the second-order condition.

$\rightarrow$ Given $\pi_K = \frac{1}{4} L^{\frac{1}{2}} K^{-\frac{3}{4}} - r$ and $\pi_L = \frac{1}{2} K^{\frac{1}{4}} L^{-\frac{1}{2}} - w$

$\pi_{KK} = -\frac{3}{16} K^{-\frac{7}{4}} L^{\frac{1}{2}}$ | $\pi_{LK} = \frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}}$

$\pi_{KL} = \frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}}$ | $\pi_{LL} = -\frac{1}{4} K^{\frac{1}{4}} K^{-\frac{3}{2}}$

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix}$$

- $|H_1| = -\frac{3}{16} K^{-\frac{7}{4}} L^{\frac{1}{2}}$

$\left(\frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}}\right)\left(\frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}}\right)$

- $|H_2| = \begin{vmatrix} -\frac{3}{16} K^{-\frac{7}{4}} L^{\frac{1}{2}} & \frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}} \\ \frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}} & -\frac{1}{4} K^{\frac{1}{4}} K^{-\frac{3}{2}} \end{vmatrix}$

$\left(-\frac{3}{16} K^{-\frac{7}{4}} L^{\frac{1}{2}}\right)\left(-\frac{1}{4} K^{\frac{1}{4}} K^{-\frac{3}{2}}\right)$

$$|H_2| = \left(-\frac{3}{16} K^{-\frac{7}{4}} L^{\frac{1}{2}}\right)\left(-\frac{1}{4} K^{\frac{1}{4}} K^{-\frac{3}{2}}\right) - \left(\frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}}\right)\left(\frac{1}{8} K^{-\frac{3}{4}} L^{-\frac{1}{2}}\right)$$

$$= \frac{3}{64} K^{-\frac{3}{2}} L^{-1} - \frac{1}{64} K^{-\frac{3}{2}} L^{-1}$$

$$= \frac{1}{32} K^{-\frac{3}{2}} L^{-1}$$

$\therefore$ $H_1 < 0$ and $H_2 > 0$: negative definite
and $d^2\pi$ is zero means it is a globally concave.

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